Exam 1 next Fri Sept 30 in class (Ungar 411).

Fund Thm, Row / Column spaces NOT on Exam 1.

HW 2 Discussion.

Problem 2:

$$R_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad \text{rotate}$$

$$F_t = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \quad \text{reflect}$$

$$P_t = \begin{pmatrix} \cos^2 t & \cos t \sin t \\ \cos t \sin t & \sin^2 t \end{pmatrix} \quad \text{project.}$$

$$R_s R_t = R_{s+t}$$ ✓

Function composition

[Equivalent to Euler e^{it}:

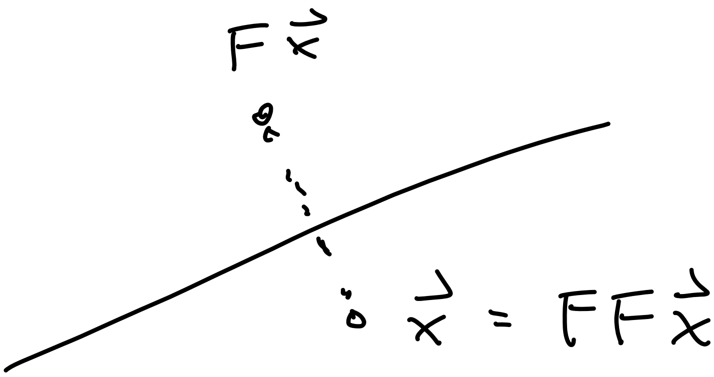
$$i \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$e^{it} = \exp\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = R_t$$

$$F_t^2 = I$$

Reflect Twice $\equiv$ Do Nothing.

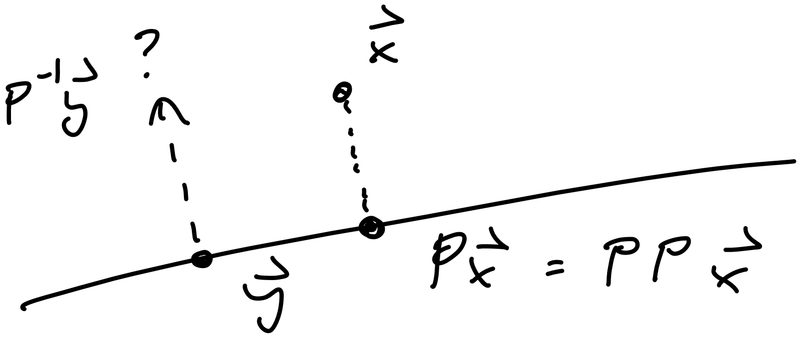
$F_t^{-1} = F_t$



$$P_t^2 = P_t$$

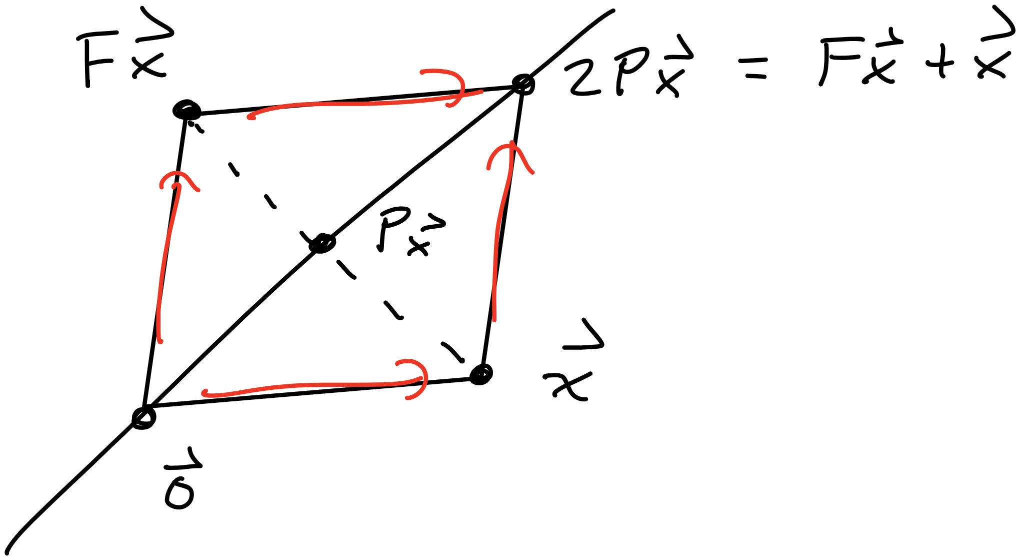
P_t^{-1} does not exist.

Project Twice $\equiv$ Project Once.



$$F_{2t} + I = 2P_t$$

reflect across line angle t.



$$2P\vec{x} = F\vec{x} + \vec{x}$$

$$2P\vec{x} = (F+I)\vec{x} \quad \forall \vec{x}$$

$$2P = F + I \quad \checkmark$$

Problem 3. The orthogonal group

$$O_n(\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : A^{-1} = A^T\}$$

called "orthogonal matrices"

Why? $A = \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{pmatrix}.$

$$A^{-1} = A^T \iff A^T A = I$$

$$\begin{pmatrix} - & \vec{a}_1^{\,T} & - \\ & \vdots & \\ - & \vec{a}_n^{\,T} & - \end{pmatrix} \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$\vec{a}_i^{\,T} \vec{a}_j = \delta_{ij}$$

$\iff$ vectors $\vec{a}_1, \ldots, \vec{a}_n$ are orthonormal.

$A \in O_n(\mathbb{R}) \iff$ columns are o.n. basis of $\mathbb{R}^n$

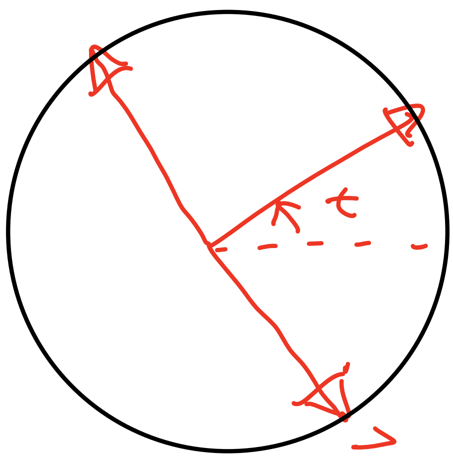
Fund Theorem. $\iff$ rows are o.n. basis of $\mathbb{R}^n$.

$$[\, A^T A = I \iff A A^T = I \,]$$

Fully describe $O_2(\mathbb{R})$:

$$A = \underbrace{\begin{pmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{pmatrix}}_{\text{orthonormal in } \mathbb{R}^2} \in O_2(\mathbb{R}).$$

$$\vec{a}_2 = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$



$$\vec{a}_1 = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$

Conclusion:

$$A \in O_2(\mathbb{R}) \implies A = R_t$$
$$\text{or } A = F_t.$$

<u>Group?</u>

① $I \in O_2(\mathbb{R})$

② $A \in O_2(\mathbb{R}) \implies A^{-1} \in O_2(\mathbb{R})$

③ $A, B \in O_2(\mathbb{R}) \Longrightarrow AB \in O_2(\mathbb{R})$

① $I^{-1} = I^T$ ✓

② $A^{-1} = A^T \overset{?}{\Longrightarrow} (A^{-1})^{-1} = (A^{-1})^T$

$(A^{-1})^{-1} = A = (A^T)^T = (A^{-1})^T$ ✓

③ $A^{-1} = A^T$ & $B^{-1} = B^T$

$(AB)^{-1} = (AB)^T$?

$(AB)^{-1} = B^{-1}A^{-1}$

$= B^T A^T = (AB)^T$ ✓

[In higher dimensions, still

$O_n(\mathbb{R})$ = essentially just rotations & reflections.

Related to Gram-Schmidt orthogonalization.

Basis of $\mathbb{R}^n \rightsquigarrow$ O.N. basis.]

Particle physicists like matrix groups.

$U(1)$ electromagnetism

$SU(2)$ weak force

$SU(3)$ strong force

Frobenius Norm:

$$A = \ell\left\{\underbrace{\begin{pmatrix} | & & | \\ \vec{a}_1 & & \vec{a}_m \\ | & & | \end{pmatrix}}_{m}\right. \qquad \vec{a}_i \in \mathbb{C}^{\ell}$$

$$B = m\left\{\underbrace{\begin{pmatrix} — & \vec{b}_1^{\,T} & — \\ & \vdots & \\ — & \vec{b}_m^{\,T} & — \end{pmatrix}}_{n}\right. \qquad \vec{b}_i \in \mathbb{C}^{n}.$$

$$AB = \sum_{i=1}^{m} \vec{a}_i \vec{b}_i^{\,T}$$

$\ell \times n$ matrix

$$\|AB\|_F = \left\| \sum \vec{a}_i \vec{b}_i^T \right\|_F$$

$$\leq \sum \|\vec{a}_i \vec{b}_i^T\|_F$$

[Think $\mathbb{C}^{\ell \times n} = \mathbb{C}^{\ell n \times 1}$]

$$= \sum \|\vec{a}_i\| \|b_i\| \qquad (b)$$

$$\leq \sqrt{\|\vec{a}_1\|^2 + \cdots + \|\vec{a}_m\|^2} \cdot \sqrt{\|\vec{b}_1\|^2 + \cdots + \|b_m\|^2} \qquad (c)$$

[Apply C.S. to real vectors $(\|\vec{a}_1\|, \dots, \|\vec{a}_m\|)$ & $(\|\vec{b}_1\|, \dots, \|\vec{b}_m\|)$]

$$= \|A\|_F \|B\|_F \qquad (a).$$

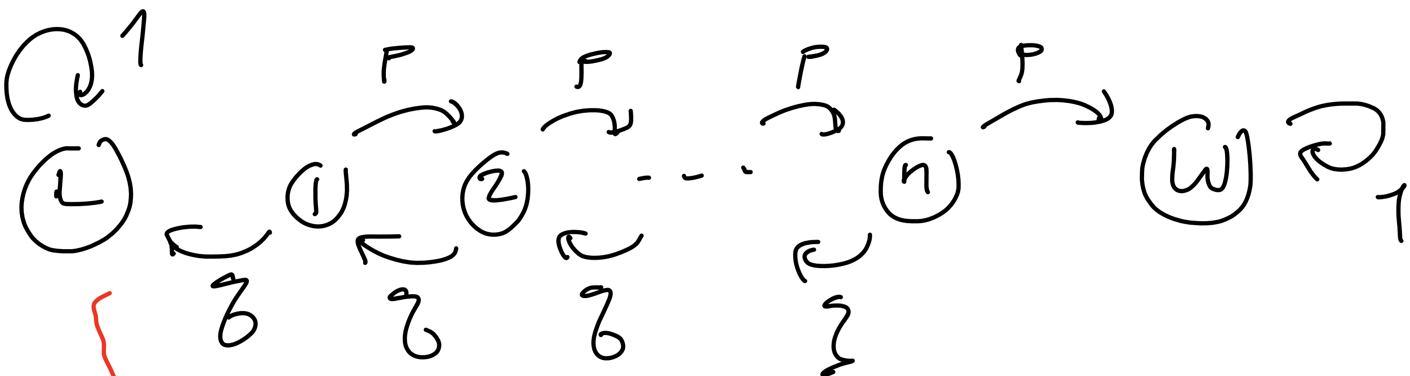
Application. Square A

$$\|A\|_F < 1 \implies (I-A)^{-1} \text{ exists}$$

$$(I-A)^{-1} = I + A + A^2 + \cdots$$

$$\left[\frac{1}{1-\lambda} = 1 + \lambda + \lambda^2 + \cdots \right]$$

Gambler's Ruin $(p+q=1)$



$$\left(\begin{array}{c|c} I & R \\ \hline 0 & Q \end{array} \right)^n \to \left(\begin{array}{c|c} I & R(I-Q)^{-1} \\ \hline 0 & 0 \end{array} \right)$$

Immediately solves the problem