Complex Numbers:

$$\mathbb{C} = \{ a + ib : a, b \in \mathbb{R} \}$$

Complex conjugation

$$\alpha \longmapsto \overline{\alpha}$$
$$a+ib \longmapsto a - ib$$

Note: $\alpha = \overline{\alpha} \iff \alpha \in \mathbb{R}$

Important: $\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$.

Absolute value

$$|\alpha| = |a+ib| = \sqrt{a^2+b^2} \geq 0.$$
$$|\alpha| = 0 \iff a = b = 0$$
$$\iff \alpha = 0.$$

Fundamental Identity

$$\overline{\alpha}\alpha = |\alpha|^2$$

$$\alpha^{-1} = \frac{\overline{\alpha}}{|\alpha|^2} \qquad (\alpha \neq 0)$$

Hermitian Space:

$$\mathbb{C}^n = \{(\alpha_1, \dots, \alpha_n) : \alpha_i \in \mathbb{C}\ \forall i\}$$

Instead of dot product, use "Hermitian inner product":

$$\langle \vec{u}, \vec{v} \rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \dots + \bar{u}_n v_n$$

$$\langle \vec{v}, \vec{u} \rangle = \bar{v}_1 u_1 + \bar{v}_2 u_2 + \dots + \bar{v}_n u_n$$

$$= \overline{\bar{u}_1 v_1} + \overline{\bar{u}_2 v_2} + \dots + \overline{\bar{u}_n v_n}$$

$$= \overline{\bar{u}_1 v_1 + \dots + \bar{u}_n v_n}$$

$$= \overline{\langle \vec{u}, \vec{v} \rangle}$$

[Alternate: $\alpha^* := \bar{\alpha}$]

$$\langle \vec{v}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{v} \rangle}$$

$$\implies \langle \vec{v}, \vec{v} \rangle \in \mathbb{R}.$$

In fact,

$$\langle \vec{v}, \vec{v} \rangle = \overline{v_1} v_1 + \cdots + \overline{v_n} v_n$$

$$= |v_1|^2 + \cdots + |v_n|^2 \geq 0.$$

$$\langle \vec{v}, \vec{v} \rangle = 0$$

$$\Longleftrightarrow |v_1| = \cdots = |v_n| = 0$$

$$\Longleftrightarrow v_1 = \cdots = v_n = 0$$

$$\Longleftrightarrow \vec{v} = \vec{0}.$$

[If you define

$$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + \cdots + u_n v_n$$

then you do not get this]

This allows us to define a norm:

$$\| \vec{v} \| = \sqrt{\langle \vec{v}, \vec{v} \rangle} \geq 0.$$

Still have Cauchy-Schwarz

$$|\langle \vec{u}, \vec{v} \rangle|^2 \leq \langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle$$

so we still have triangle ineq

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|.$$

So we still have a metric

$$\text{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|.$$

Why complex numbers?

Complex numbers simplify Fourier analysis.

$$f(x) = a_0 + \sum_{n\geq 1} a_n \sin(2\pi n x) + \sum_{n \geq 1} b_n \cos(2\pi n x)$$

$$\Downarrow$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i 2\pi n x}$$

Explanation: $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = -\frac{i}{2}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$a_n \sin(2\pi n x) = -\frac{i}{2}\left(a_n e^{i2\pi n x} - a_n e^{-i2\pi n x}\right)$$

$$b_n \cos(2\pi n x) = \frac{1}{2}\left(b_n e^{i2\pi n x} + b_n e^{-i2\pi n x}\right)$$

$$a_n \sin(2\pi n x) + b_n \cos(2\pi n x)$$

$$= \underbrace{\left(\frac{b_n - i a_n}{2}\right)}_{C_n} e^{i2\pi n x} + \underbrace{\left(\frac{b_n + i a_n}{2}\right)}_{C_{-n}} e^{-i2\pi n x}$$

Also: $C_0 = a_0$

$$a_0 + \sum a_n \sin(2\pi n x) + \sum b_n \cos(2\pi n x) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi n x}$$

Why is this better?

Only one formula:

$$\langle e^{i2\pi mx}, e^{i2\pi nx} \rangle$$

$$= \int_0^1 \overline{e^{i2\pi mx}}\, e^{i2\pi nx}\, dx$$

$$= \int_0^1 e^{-i2\pi mx}\, e^{i2\pi nx}\, dx$$

$$= \int_0^1 e^0\, dx = 1 \quad \text{if } m=n.$$

else

$$= \int_0^1 e^{i2\pi(n-m)x}\, dx$$

$$= \frac{1}{i2\pi(n-m)}\, e^{i2\pi(n-m)x} \Big|_0^1$$

$$= \frac{1}{i2\pi(n-m)}\left[1-1\right] = 0$$

✓

Need to redefine

$L^2[0,1]$ = functs $[0,1] \to \mathbb{C}$

satisfying $\int_0^1 |f(x)|^2 dx < \infty$

Hermitian inner prod

$$\langle f(x), g(x) \rangle = \int_0^1 \overline{f(x)}\, g(x)\, dx$$

Suppose we can write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx}$$

Then using $\langle e^{i2\pi mx}, e^{i2\pi nx} \rangle = \delta_{mn}$

and HW 1.3 we get

$$c_n = \langle f(x), e^{i2\pi nx} \rangle$$

$$= \int_0^1 \overline{f(x)}\, e^{i2\pi nx}\, dx$$

If $f(x) \in \mathbb{R}$

then $\overline{f(x)} = f(x)$.

One last step. What about

$$L^2(\mathbb{R}) = \text{funcs } f : \mathbb{R} \to \mathbb{R}$$

$$\int_{-\infty}^{\infty} f(x)^2\, dx < \infty$$

Problem: Fourier series fail!

$$f(x) \neq \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx}$$

$$f(x) = \int_{t=-\infty}^{\infty} c_t e^{i2\pi tx}\, dt$$

This is called the <u>Fourier transform</u>.