

Complex Numbers:

$$\mathbb{C} = \{ a + ib : a, b \in \mathbb{R} \}$$

Complex conjugation

$$\begin{aligned} \alpha &\mapsto \bar{\alpha} \\ a + ib &\mapsto a - ib \end{aligned}$$

Note: $\alpha = \bar{\alpha} \iff \alpha \in \mathbb{R}$

Important: $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$.

Absolute value

$$|\alpha| = |a + ib| = \sqrt{a^2 + b^2} \geq 0.$$

$$|\alpha| = 0 \iff a = b = 0$$

$$\iff \alpha = 0.$$

Fundamental Identity

$$\bar{\alpha}\alpha = |\alpha|^2$$

$$\alpha^{-1} = \frac{\bar{\alpha}}{|\alpha|^2} \quad (\alpha \neq 0)$$



Hermitian Space :

$$\mathbb{C}^n = \{ (\alpha_1, \dots, \alpha_n) : \alpha_i \in \mathbb{C} \forall i \}$$

Instead of dot product, use

"Hermitian inner product" :

$$\langle \vec{u}, \vec{v} \rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \dots + \bar{u}_n v_n$$

$$\langle \vec{v}, \vec{u} \rangle = \bar{v}_1 u_1 + \bar{v}_2 u_2 + \dots + \bar{v}_n u_n$$

$$= \overline{\bar{u}_1 v_1 + \bar{u}_2 v_2 + \dots + \bar{u}_n v_n}$$

$$= \overline{\langle \vec{u}, \vec{v} \rangle}$$

$$= \langle \vec{u}, \vec{v} \rangle$$

[Alternate: $\alpha^* := \bar{\alpha}$]

$$\langle \vec{v}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{v} \rangle}$$

$$\implies \langle \vec{v}, \vec{v} \rangle \in \mathbb{R}.$$

In fact,

$$\begin{aligned} \langle \vec{v}, \vec{v} \rangle &= \bar{v}_1 v_1 + \dots + \bar{v}_n v_n \\ &= |v_1|^2 + \dots + |v_n|^2 \geq 0. \end{aligned}$$

$$\langle \vec{v}, \vec{v} \rangle = 0$$

$$\Leftrightarrow |v_1| = \dots = |v_n| = 0$$

$$\Leftrightarrow v_1 = \dots = v_n = 0$$

$$\Leftrightarrow \vec{v} = \vec{0}.$$

[If you define $\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + \dots + u_n v_n$ then you do not get this]

This allows us to define a norm:

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} \geq 0.$$

Still have Cauchy-Schwarz

$$|\langle \vec{u}, \vec{v} \rangle|^2 \leq \langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle$$

so we still have triangle inequality

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|.$$

So we still have a metric

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$



Why complex numbers?

Complex numbers simplify Fourier analysis.

$$f(x) = a_0 + \sum_{n \geq 1} a_n \sin(2\pi n x)$$

$$+ \sum_{n \geq 1} b_n \cos(2\pi n x)$$



$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n x}$$

Explanation:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

$$\cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta})$$

$$\sin \theta = -\frac{i}{2} (e^{i\theta} - e^{-i\theta})$$

↓

$$a_n \sin(2\pi n x) = -\frac{i}{2} (a_n e^{i2\pi n x} - a_n e^{-i2\pi n x})$$

$$b_n \cos(2\pi n x) = \frac{1}{2} (b_n e^{i2\pi n x} + b_n e^{-i2\pi n x})$$

↓

$$a_n \sin(2\pi n x) + b_n \cos(2\pi n x)$$

$$= \left(\frac{b_n - i a_n}{2} \right) e^{i2\pi n x} + \left(\frac{b_n + i a_n}{2} \right) e^{-i2\pi n x}$$

C_n C_{-n}

Also: $C_0 = a_0$

↓

$$a_0 + \sum a_n \sin(2\pi n x) + \sum b_n \cos(2\pi n x) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi n x}$$

Why is this better?

only one formula:

$$\left\langle e^{i2\pi m x}, e^{i2\pi n x} \right\rangle$$

$$= \int_0^1 \frac{1}{e^{i2\pi m x}} e^{i2\pi n x} dx$$

$$= \int_0^1 e^{-i2\pi m x} e^{i2\pi n x} dx$$

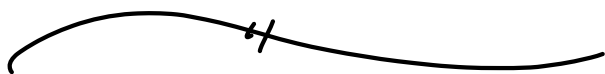
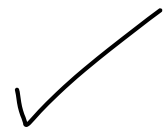
$$= \int_0^1 e^0 dx = 1 \quad \text{if } m=n,$$

else

$$= \int_0^1 e^{i2\pi(n-m)x} dx$$

$$= \frac{1}{i2\pi(n-m)} e^{i2\pi(n-m)x} \Big|_0^1$$

$$= \frac{1}{i2\pi(n-m)} [1 - 1] = 0$$



Need to redefine

$$L^2[0,1] = \text{funct's } [0,1] \rightarrow \mathbb{C}$$

$$\text{satisfying } \int_0^1 |f(x)|^2 dx < \infty$$

Hermitean inner prod

$$\langle f(x), g(x) \rangle = \int_0^1 \overline{f(x)} g(x) dx$$



Suppose we can write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n x}$$

$$\text{Then using } \langle e^{i2\pi m x}, e^{i2\pi n x} \rangle = \delta_{mn}$$

and HW 1.3 we get

$$c_n = \langle f(x), e^{i2\pi nx} \rangle$$

$$= \int_0^1 \overline{f(x)} e^{i2\pi nx} dx$$

IF $f(x) \in \mathbb{R}$
then $\overline{f(x)} = f(x)$.



One last step. What about

$$L^2(\mathbb{R}) = \text{Funcs } f: \mathbb{R} \rightarrow \mathbb{R} \\ \int_{-\infty}^{\infty} f(x)^2 dx < \infty$$

Problem: Fourier series fail!

$$f(x) \neq \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx}$$

$$f(x) = \int_{t=-\infty}^{\infty} c_t e^{i2\pi t x} dt$$

This is called the Fourier
transform.