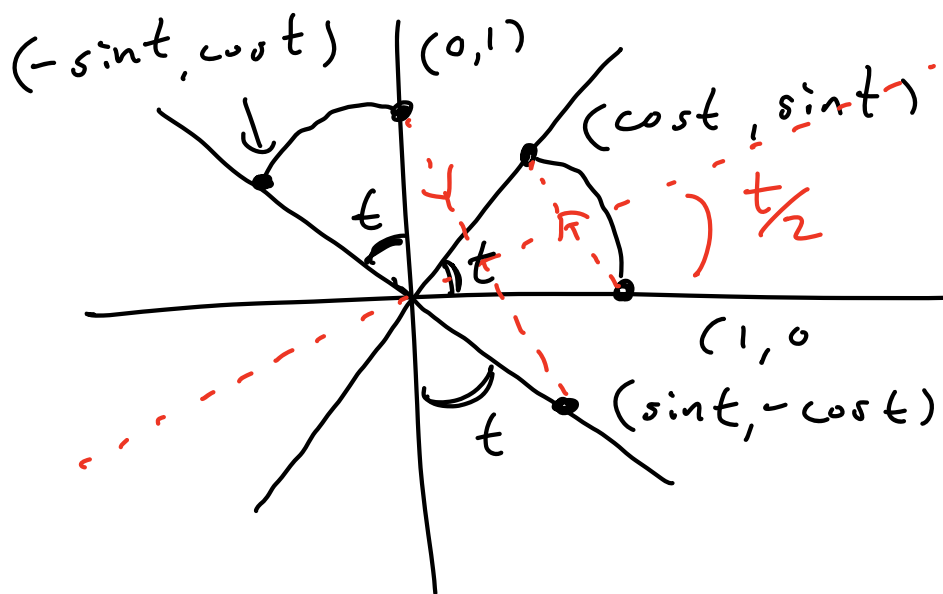


HW 2 due Friday.

Special 2×2 matrices.

R_t rotation c.c.w. by angle t .

$$F_t = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$



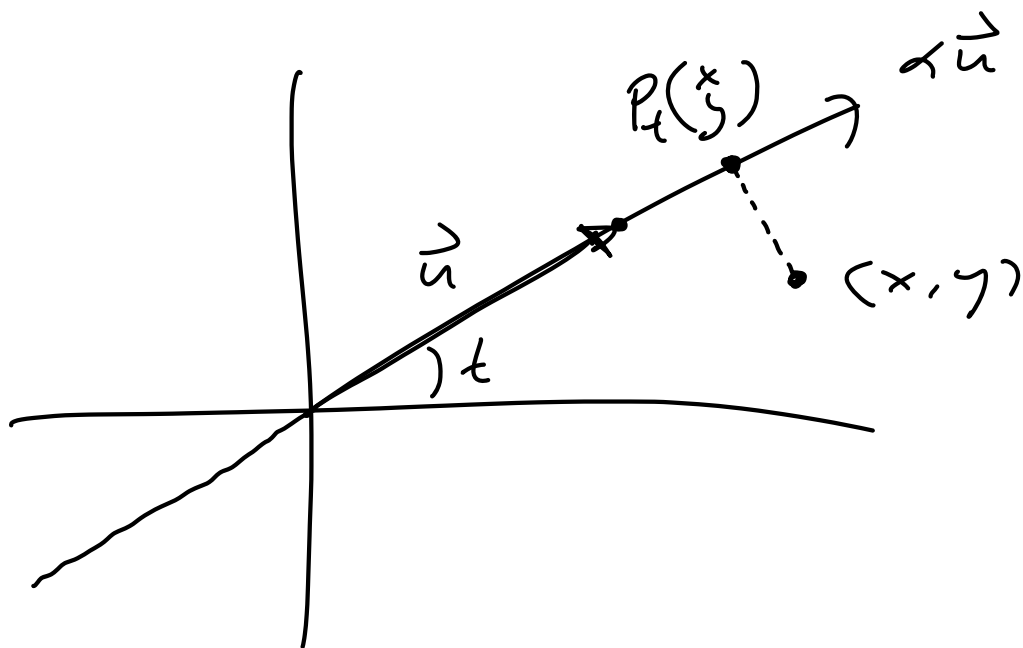
Reflect
across
this line
angle $t/2$ from
x-axis

$$P_t = \begin{pmatrix} \cos^2 t & \cos t \sin t \\ \cos t \sin t & \sin^2 t \end{pmatrix}$$

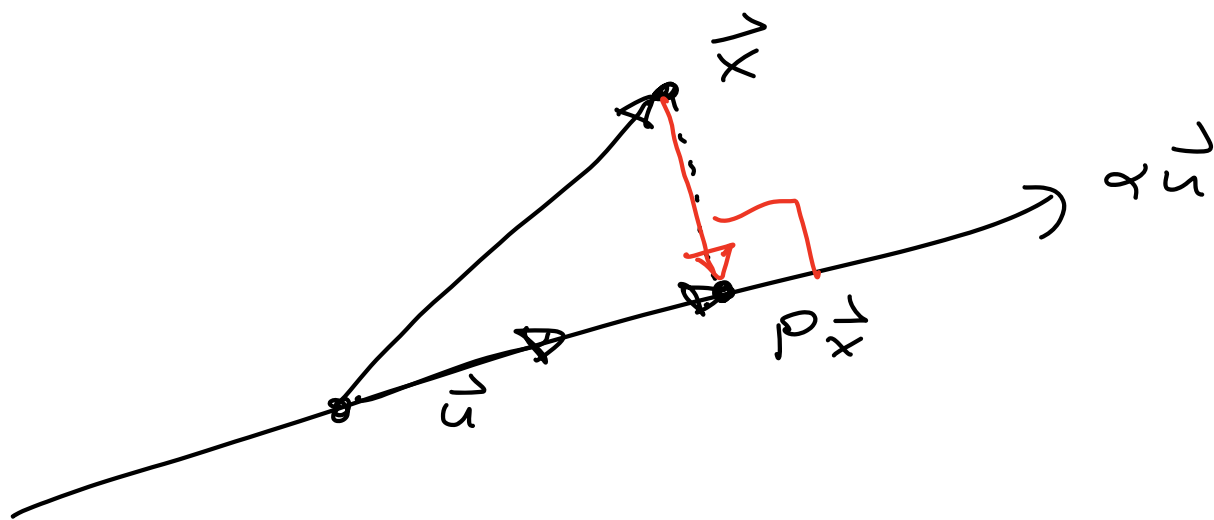
$$= \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \begin{pmatrix} \cos t & \sin t \end{pmatrix}$$

Write $\vec{u} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$.

Claim: P_t projects onto the line $\alpha \vec{u}$, $\alpha \in \mathbb{R}$.



Projection in general:



$P = ?$

We know exactly two things:

$$\textcircled{1} \quad P\vec{x} = \alpha\vec{u} \text{ for some } \alpha.$$

$$\textcircled{2} \quad P\vec{x} - \vec{x} \perp \vec{u}.$$

Solve for P (P is a matrix).

Combine $\textcircled{1}$ & $\textcircled{2}$.

$$P\vec{x} - \vec{x} \perp \vec{u} \quad \text{dot product.}$$

$$\vec{u}^T (P\vec{x} - \vec{x}) = 0.$$

$$\vec{u}^T (\alpha\vec{u} - \vec{x}) = 0$$

$$\alpha \vec{u}^T \vec{u} - \vec{u}^T \vec{x} = 0$$

$$\alpha \vec{u}^T \vec{u} = \vec{u}^T \vec{x}$$

$$\alpha = \vec{u}^T \vec{x} / \vec{u}^T \vec{u}$$

$$\|\vec{u}\|^2 = 1.$$

$$\alpha = \vec{u}^T \vec{x}$$

dot product.

To project \vec{x} onto the line $\alpha\vec{u}$.

$$P\vec{x} = \alpha\vec{u} = \underbrace{(\vec{u}^T \vec{x})}_{\text{scalar}} \underbrace{\vec{u}}_{\text{vector}}$$

Now the miracle of associativity:

$$\begin{aligned} P\vec{x} &= (\vec{u}^T \vec{x}) \vec{u} \\ &= \vec{u} (\vec{u}^T \vec{x}) \quad \text{just a scalar} \\ &= (\underbrace{\vec{u} \vec{u}^T}_{\text{matrix!}}) \underbrace{\vec{x}}_{\text{vector}} \quad \text{MAGIC!} \end{aligned}$$

Conclusion: The matrix that projects onto the line $\alpha\vec{u}$ (say $\|\vec{u}\|=1$) is

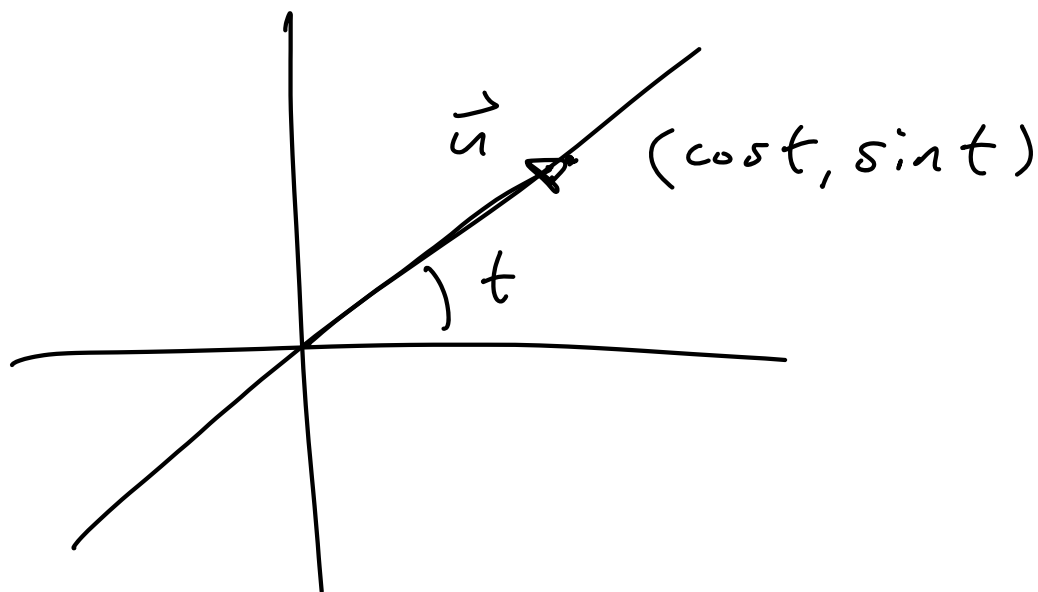
$$P = \vec{u} \vec{u}^T$$

Example: Project onto x-axis
(i.e. line $\propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$).

$$P = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} \checkmark$$

Project onto line of angle t :



$$P = \vec{u} \vec{u}^T = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} (\cos t \ \sin t)$$

Look Ahead ...

Consider matrix A $m \times n$

Column space $C(A)$ = subspace
of \mathbb{R}^m spanned by cols of A .

The matrix that projects onto
the column space of A is

$$P = A(A^T A)^{-1} A^T$$

BIG THEOREMS :

Given matrix A ,

• IF A^{-1} exists then A is SQUARE.

• For square A & B ,

$$AB = I \iff BA = I.$$

These follow from the "Fundamental Theorem of Linear algebra":

Given $m \times n$ matrix A , define

row space $R(A) \subseteq \mathbb{R}^n$

column space $C(A) \subseteq \mathbb{R}^m$.

Then

$$\dim R(A) = \dim C(A)$$

& this common dimension is called the rank of A .

The idea of the proof :

Given invertible $m \times m$ E
 $n \times n$ F

then $\dim R(A) = \dim R(EAF)$.

$\dim C(A) = \dim C(EAF)$.

Then to prove the theorem, we
look for invertible matrices E & F
such that

$$EAF = \begin{pmatrix} I_r & O_{r \times (n-r)} \\ O_{(m-r) \times r} & O_{(m-r) \times (n-r)} \end{pmatrix}$$

The matrix is partitioned into four quadrants. A horizontal brace above the matrix spans all columns and is labeled n . A vertical brace to the right of the top two rows is labeled r . A vertical brace to the right of the bottom two rows is labeled $m-r$. A horizontal brace below the first r columns is labeled r . A horizontal brace below the last $n-r$ columns is labeled $n-r$. The top-left quadrant is labeled I_r , the top-right is $O_{r \times (n-r)}$, the bottom-left is $O_{(m-r) \times r}$, and the bottom-right is $O_{(m-r) \times (n-r)}$. A large left brace is labeled m .

Performs elementary row operations.

Performs elementary col operations.

Then easy to see

$$\dim C(EAF) = \dim R(EAF) = r.$$