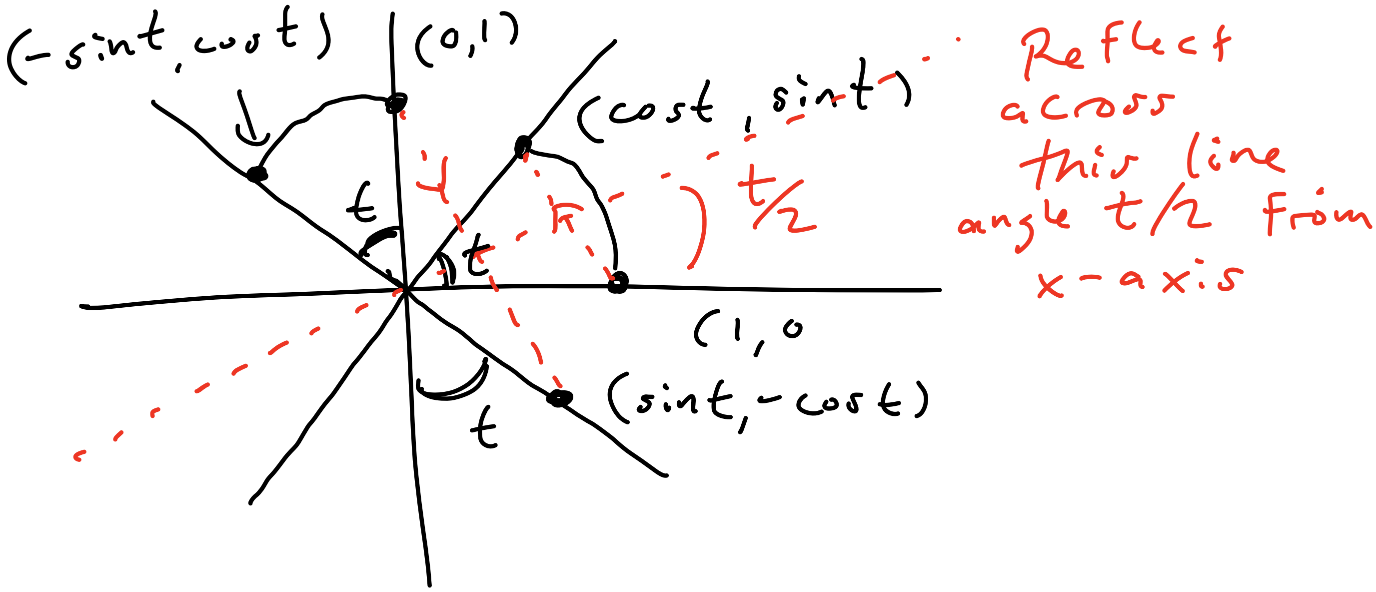
HW 2 due Friday.

Special 2×2 matrices.

R_t rotation c.c.w. by angle t.

$$F_t = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

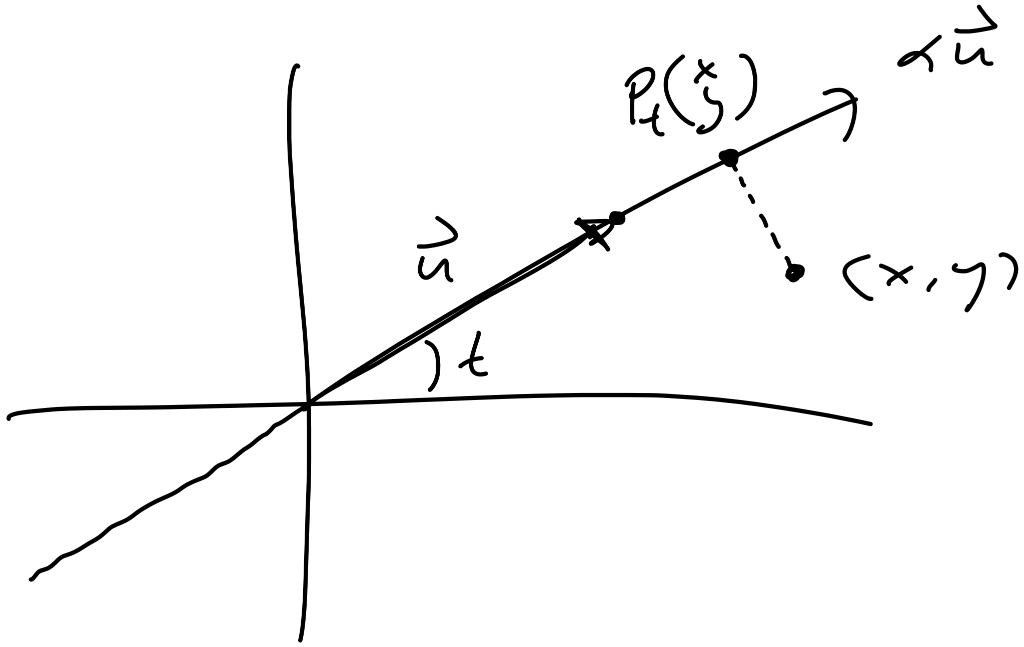


$$P_t = \begin{pmatrix} \cos^2 t & \cos t \sin t \\ \cos t \sin t & \sin^2 t \end{pmatrix}.$$

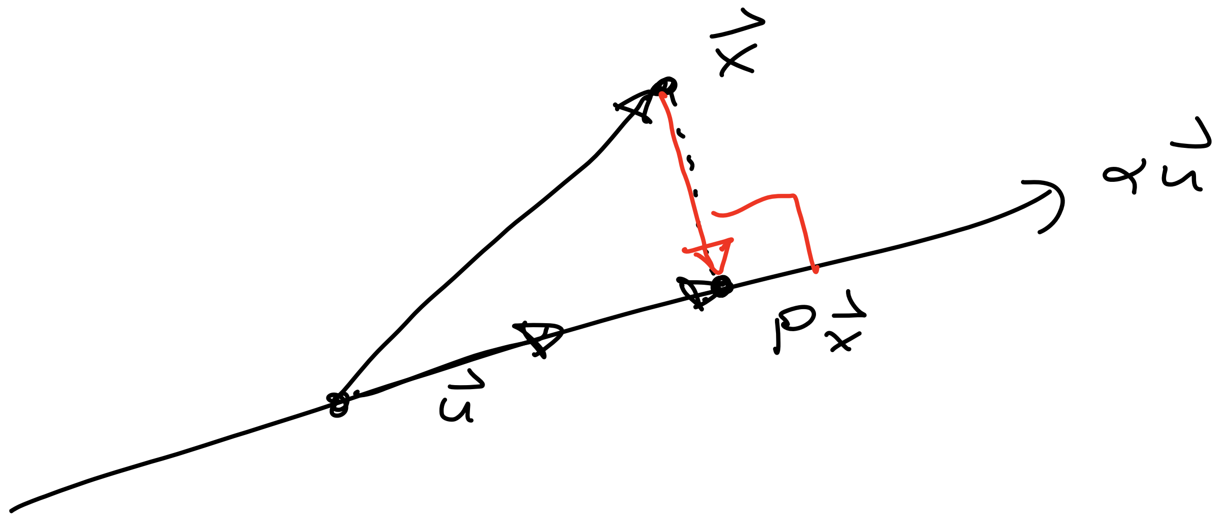
$$= \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \begin{pmatrix} \cos t & \sin t \end{pmatrix}$$

Write $\vec{u} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$.

Claim: P_t projects onto the line $\alpha\vec{u}$, $\alpha \in \mathbb{R}$.



Projection in general:



$P = ?$

We know exactly two things:

① $P\vec{x} = \alpha \vec{u}$ for some α.

② $P\vec{x} - \vec{x} \perp \vec{u}$.

Solve for P (P is a matrix).

Combine ① & ②.

$$P\vec{x} - \vec{x} \perp \vec{u}$$

dot product.

$$\vec{u}^T (P\vec{x} - \vec{x}) = 0.$$

$$\vec{u}^T (\alpha \vec{u} - \vec{x}) = 0$$

$$\alpha \vec{u}^T \vec{u} - \vec{u}^T \vec{x} = 0$$

$$\alpha \vec{u}^T \vec{u} = \vec{u}^T \vec{x}$$

$$\alpha = \vec{u}^T \vec{x} \Big/ \overset{1}{\vec{u}^T \vec{u}}$$

$\|\vec{u}\|^2 = 1.$

$$\alpha = \vec{u}^T \vec{x}$$

dot product.

To project $\vec{x}$ onto the line $\alpha \vec{u}$.

$$P\vec{x} = \alpha \vec{u} = \underbrace{(\vec{u}^T \vec{x})}_{\text{scalar}} \underbrace{\vec{u}}_{\text{vector}}$$

Now the miracle of associativity:

$$P\vec{x} = (\vec{u}^T\vec{x})\,\vec{u}$$

$$= \vec{u}\,(\vec{u}^T\vec{x})$$

just a scalar

MAGIC!

$$= \underbrace{(\vec{u}\,\vec{u}^T)}_{\text{matrix!}}\,\vec{x}$$

vector.

Conclusion: The matrix that projects onto the line $\alpha\vec{u}$ (say $\|\vec{u}\| = 1$) is

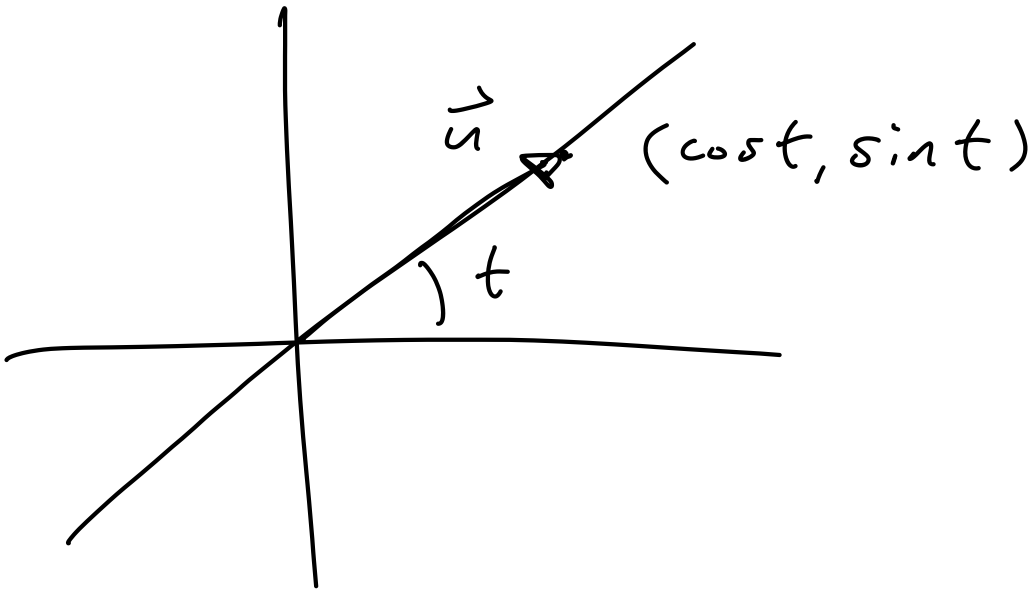
$$P = \vec{u}\,\vec{u}^T.$$

Example: Project onto x-axis (i.e. line $\alpha\begin{pmatrix}1\\0\end{pmatrix}$).

$$P = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$P\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix} \checkmark$$

Project onto line of angle t:



$$P = \vec{u}\,\vec{u}^T = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} (\cos t \;\; \sin t)$$

✓

Look Ahead ...

Consider matrix A $m \times n$

Column space $C(A)$ = subspace of $\mathbb{R}^m$ spanned by cols of A.

The matrix that projects onto the column space of A is

$$P = A(A^TA)^{-1}A^T$$

BIG THEOREMS :

Given matrix A,

- If A^{-1} exists then A is SQUARE.
- For square A & B,

$$AB = I \iff BA = I.$$

These follow from the "Fundamental Theorem of Linear algebra":

Given $m \times n$ matrix A, define

row space $R(A) \subseteq \mathbb{R}^n$
column space $C(A) \subseteq \mathbb{R}^m$.

Then

$$\dim R(A) = \dim C(A)$$

& this common dimension is called <u>the rank</u> of A.

The idea of the proof:

Given <u>invertible</u> $m \times m$ E

$n \times n$ F

then $\dim R(A) = \dim R(EAF)$.

$\dim C(A) = \dim C(EAF)$.

Then to prove the theorem, we look for invertible matrices E & F such that

$$EAF = \begin{array}{cc} & \overbrace{\qquad\qquad\qquad\qquad\qquad}^{n} \\ m\left\{\vphantom{\begin{array}{c}I\\I\\I\end{array}}\right. & \left(\begin{array}{c|c} I_r & O_{r\times n-r} \\ \hline O_{m-r\times r} & O_{m-r\times n-r}\end{array}\right)\begin{array}{l}\}\,r\\ \}\,m-r\end{array} \\ & \underbrace{\qquad\quad}_{r}\;\underbrace{\qquad\quad}_{n-r}\end{array}$$

Performs elementary row operations.

Performs elementary col operations.

Then easy to see

$$\dim C(EAF) = \dim R(EAF) = r.$$