Matrix Algebra:

Let $\mathbb{R}^{m \times n}$ = set of $m \times n$ matrices with real entries.

Convention: $\mathbb{R}^{n \times 1} = \mathbb{R}^n$

Note $\mathbb{R}^{m \times n}$ is a vector space over $\mathbb{R}$ with matrix addition

$$A = (a_{ij}),\ B = (b_{ij}) \Rightarrow$$

$$A + B = (a_{ij} + b_{ij})$$

and scalar mult

$$\alpha A = \alpha (a_{ij}) = (\alpha a_{ij}).$$

Note: $\dim \mathbb{R}^{m \times n} = mn$

with basis

$$E_{k\ell} = \begin{array}{cc} & \ell \\ k & \left(\begin{array}{ccc} & 0 & \\ 0 & \vdots & 0 \\ & 0 & \\ 0 \cdots 0 & 1 & 0 \cdots 0 \\ & 0 & \\ 0 & \vdots & 0 \\ & 0 & \end{array} \right) \end{array}$$

But matrices have an extra algebraic operation:

$$\mathbb{R}^{l\times m} \times \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{l\times n}$$
$$(A\ ,\ B) \longmapsto AB$$

Also have transposition:

$$\mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{n\times m}$$
$$A \longmapsto A^T$$
$$[A \longmapsto A^* = \bar{A}^T \text{ over } \mathbb{C}]$$

[1×1 matrices are numbers.]

We also zero matrices of any shape

$$0_{m\times n} = m\left\{\underbrace{\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}_{n}\right.$$

and identity matrices of SQUARE shape:

$$I_n = n\left\{\underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}}_{n}\right.$$

Rules of Matrix Algebra:

- Vector space rules for $+$ & $\cdot$

  scalar multiplication

- $AB \neq BA$ in general

- Multiplication is bilinear:

$$A(\beta B + \gamma C) = \beta AB + \gamma AC$$

$$(\alpha A + \beta B)C = \alpha AC + \beta BC$$

- Properties of $O_{m\times n}$ & I_n :

$$\left.\begin{aligned} AO &= O\\ OA &= O\\ AI &= A\\ IA &= A\end{aligned}\right\}\ \text{whenever shapes allow}$$

Reason for the definition of I.

- Properties of Transpose :

$$(A+B)^T = A^T + B^T$$
$$(\alpha A)^T = \alpha A^T$$
$$(AB)^T = B^T A^T$$

[If A $\ell\times m$, B $m\times n$
A^T $m\times \ell$, B^T $n\times m$
then $A^T B^T$ does not exist!]

$$(A+B)^* = A^* + B^*$$
$$(\alpha A)^* = \alpha^* A^*$$
$$(AB)^* = B^* A^*.$$

Concept of a matrix norm:

$$\|\cdot\| : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$$

Must satisfy

- $\|A\| \geq 0$ & $\|A\| = 0 \iff A = O_{m\times n}$
- $\|\alpha A\| = |\alpha| \|A\|$
- $\|A+B\| \leq \|A\| + \|B\|$
- $\|AB\| \leq \|A\| \|B\|$

sub-multiplicative

Two main examples:

- Frobenius norm:

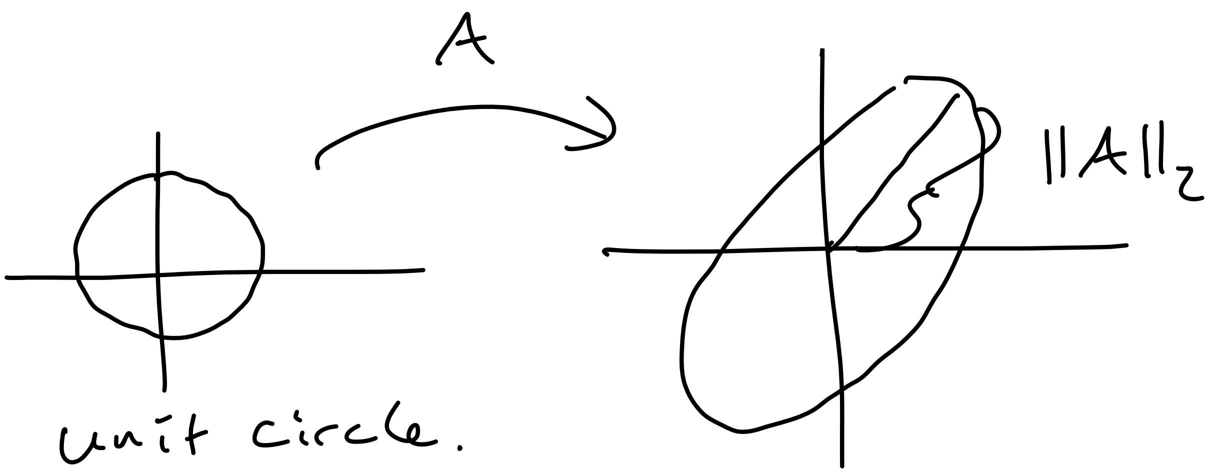
$$\|A\|_F^2 = \sum_{i,j} a_{ij}^2$$

$$= \sum_{i,j} |a_{ij}|^2 \text{ over } \mathbb{C}$$

- L^2 norm (operator norm):

$$\|A\|_2^2 = \max\left\{ \frac{\|A\vec{x}\|}{\|\vec{x}\|} : \vec{0} \neq \vec{x} \in \mathbb{R}^n \right\}$$

Picture:



Say more later ...

Recall from last time:

$m \times n$ matrix A defines a function

$$\begin{aligned} \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \vec{x} &\longmapsto A\vec{x} \end{aligned}$$

Why linear?

$$A(a\vec{x} + b\vec{y}) = aA\vec{x} + bA\vec{y}.$$

Conversely, any linear function

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

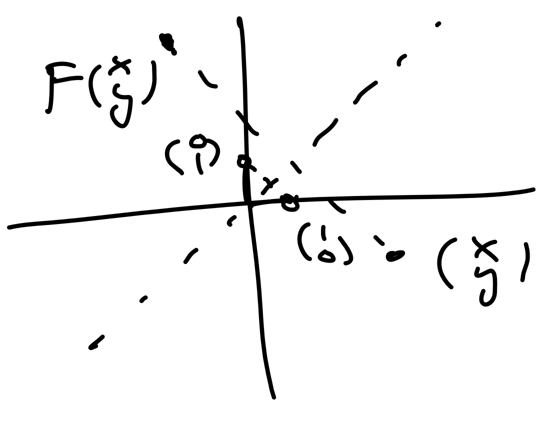
comes from a matrix. Proof:

Define $A = [T] = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}$

Then (by definition of matrix multiplication) we have

$$A\vec{x} = T(\vec{x}) \quad \forall \vec{x} \in \mathbb{R}^n.$$

For example: Reflection across the line $y = x$ in $\mathbb{R}^2$ is a linear function



The corresponding matrix is

$$[F] = \begin{bmatrix} | & | \\ F\begin{pmatrix}1\\0\end{pmatrix} & F\begin{pmatrix}0\\1\end{pmatrix} \\ | & | \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In general,

$$[F]\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

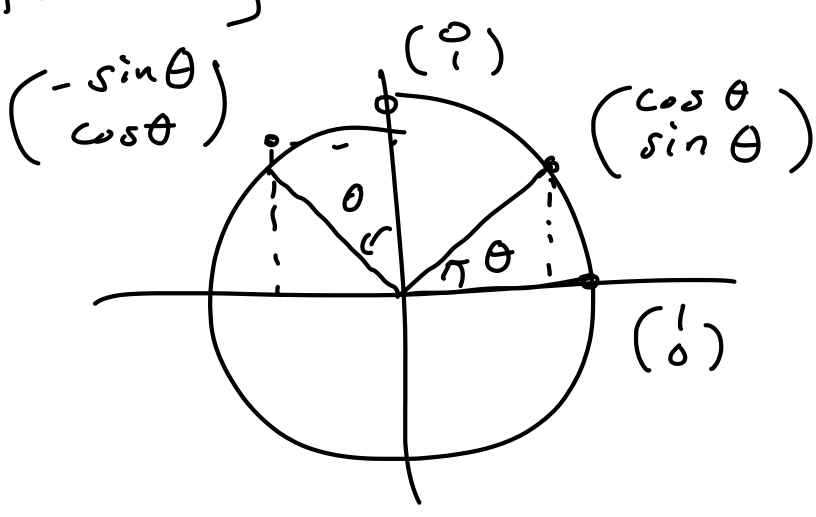
$$= \begin{pmatrix}y\\x\end{pmatrix}.$$

switches the two coords ✓

Another example:

$$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

Rotation by θ c.c.w. around $\vec{0}$.



[Remark: R_θ is linear.]

Corresponding matrix:

$$[R_\theta] = \begin{pmatrix} | & | \\ R_\theta\begin{pmatrix}1\\0\end{pmatrix} & R_\theta\begin{pmatrix}0\\1\end{pmatrix} \\ | & | \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The rotation of a <u>general</u> vector is

$$[R_\theta]\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}.$$

Interesting Linear Functions

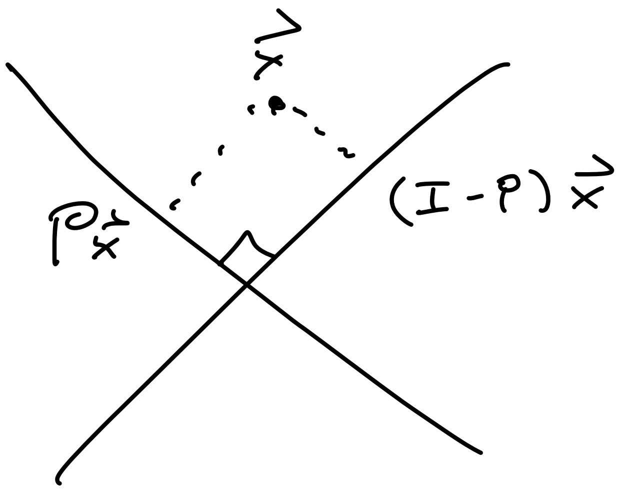
$$\mathbb{R}^n \longrightarrow \mathbb{R}^n$$

- Projection onto a subspace.
- Reflection across a subspace.
- Rotation around an $(n-2)$-dimensional subspace.

From the HW:

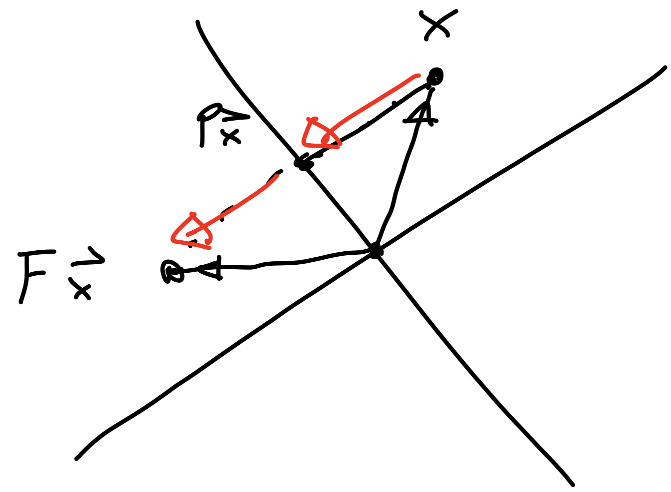
- Projection P satisfies

$$P^2 = P \quad \& \quad P(I-P) = 0$$



- Reflection F looks like.

$$F = 2P - I$$



$$F\vec{x} = \vec{x} + 2(P\vec{x} - \vec{x})$$

$$= 2P\vec{x} - \vec{x}$$

$$= (2P - I)\vec{x}$$