HW 3 due Mon Oct 17.

Reminder from last time.

Let A be $m \times n$ with rank r.

A has left inverse

$\iff r = n$

$\iff$ columns are independent.

[HW: In this case $(A^TA)^{-1}$ exists and $(A^TA)^{-1}A^T$ is one example of a left inverse.]

A has right inverse

$\iff r = m$

$\iff$ rows are ind.

[HW: In this case $A^T(AA^T)^{-1}$ is an example of a right inverse.]

Reminder : A $m \times n$ X $n \times m$

$AX = I_m$

Solve for $X = \begin{pmatrix} | & & | \\ \vec{x}_1 & \cdots & \vec{x}_m \\ | & & | \end{pmatrix}$.

$$AX = I$$

$$A(\vec{x}_1 \cdots \vec{x}_m) = (\vec{e}_1 \cdots \vec{e}_m).$$

$$(A\vec{x}_1 \cdots A\vec{x}_m) = (\vec{e}_1 \cdots \vec{e}_m)$$

$$A\vec{x}_j = \vec{e}_j \quad \forall j.$$

Note $\vec{x}_j$ exists $\iff$ $\vec{e}_j \in C(A)$.

$[\, C(A) = \{ A\vec{x} \}. \,]$

X exists $\iff$ $\vec{e}_j \in C(A)$ $\forall j$.

$$C(A) \subseteq \mathbb{R}^m$$

every basis vector is in here.

$\iff C(A) = \mathbb{R}^m$

$\iff r = m.$

A has both right and left inverse $\iff$ $m = r = n$.

Fund. Thm.

[Square with ind rows & cols].

In which case $\exists$ B, C,

$$AB = I \quad \& \quad CA = I.$$

Then

$$B = IB = CAB = CI = C.$$

What's it for?

Solving linear systems.

System of m linear equations in n unknowns:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1, \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

$$\underset{m \times n}{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}} \underset{n \times 1}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \underset{m \times 1}{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}$$

$$A\vec{x} = \vec{b}$$

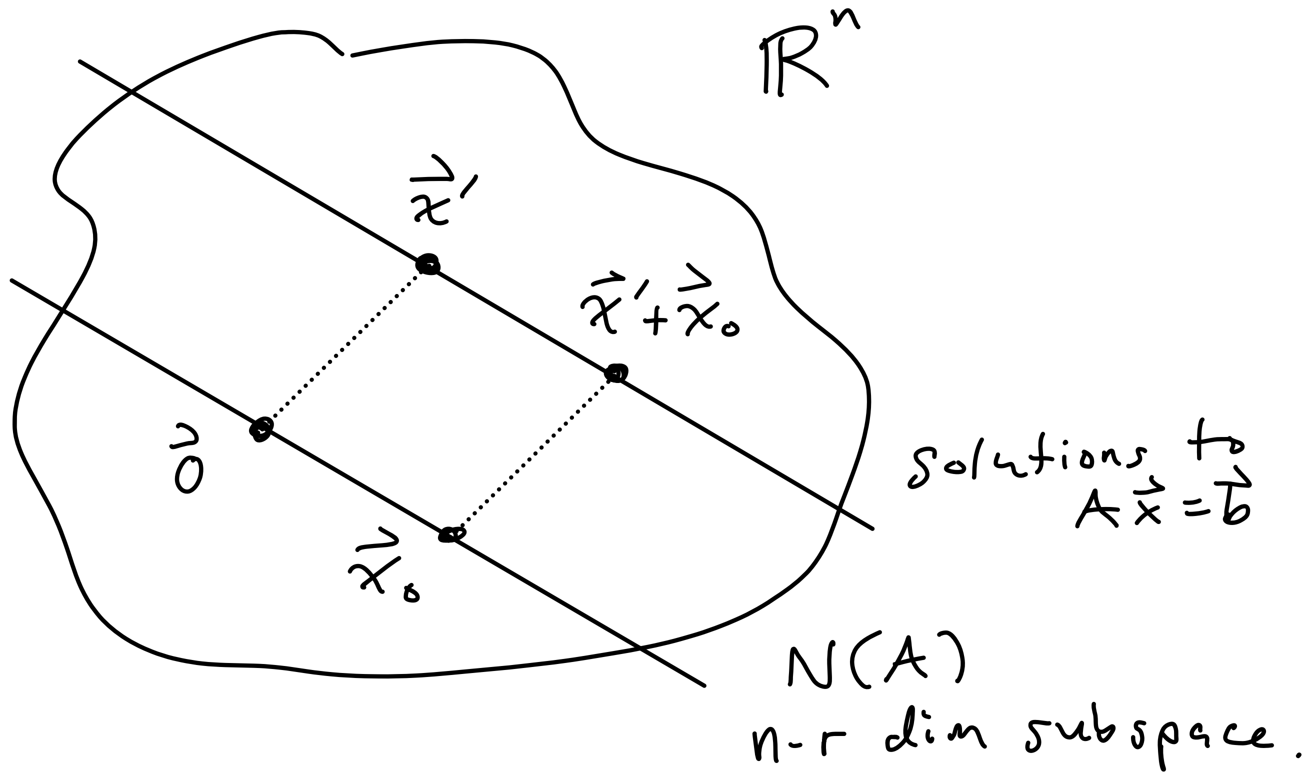
By definition:

$A\vec{x} = \vec{b}$ has (at least one) solution $\vec{x}$ $\iff$ $\vec{b} \in C(A)$.

If A^{-1} is invertible then we just solve:

$$A\vec{x} = \vec{b}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b} \quad \checkmark$$

If $\vec{b} \in C(A)$ then the solution exists and forms an $n-r$ dimensional affine subspace of $\mathbb{R}^n$:



Precise: Let $A\vec{x}' = \vec{b}$ be one particular solution. Then every solution $A\vec{x} = \vec{b}$ has form

$$\vec{x} = \underbrace{\vec{x}'}_{\text{one particular solution}} + \underbrace{\vec{x}_0}_{\text{general homogeneous sol.}}$$

for a unique $\vec{x}_0 \in N(A)$.

e.g. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

$$N(A) = \text{line} \begin{pmatrix} t \\ -2t \\ t \end{pmatrix} \subseteq \mathbb{R}^3.$$

General solution of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

is
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\underbrace{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}}_{\text{one particular solution}} + t \underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}_{\text{nullspace}}}_{\text{this is a line in 3D.}}$$

We know that a solution (x', y', z') exists because

$$C(A) = \left\{ x\begin{pmatrix}1\\1\end{pmatrix} + y\begin{pmatrix}1\\2\end{pmatrix} + z\begin{pmatrix}1\\3\end{pmatrix} \right\}$$

Fills up all of $\mathbb{R}^2$.

When $r = n$ (independent cols) then then the soln (if it exists) is unique because $N(A) = \{\vec{0}\}$.

Matrix Algebra: Suppose

$$\exists\, \vec{x}, \quad A\vec{x} = \vec{b}$$

IF $r = n$ then $(A^TA)^{-1}$ exists &

$$(A^TA)^{-1}\underbrace{A^TA}\,\vec{x} = (A^TA)^{-1}A^T\vec{b}$$

$$\vec{x} = \underbrace{(A^TA)^{-1}A^T\vec{b}}_{\text{unique.}}$$

How to find the solution:

Matrix Elimination.

Given $m \times n$ A we can find elementary $E_1, \dots, E_k$ to put A into RREF (reduced row echelon form):

$$\underbrace{E_k \cdots E_2 E_1}_{\text{does not depend on seq. of operations.}} A = \underbrace{\begin{pmatrix} 1 & * & 0 & * & * & 0 \\ & & 1 & * & * & 0 \\ & & & & & 1 \\ & & & & & \end{pmatrix}}_{\text{why this? Exists \& is Unique.}}$$

To solve $A\vec{x} = \vec{b}$ form augmented matrix $(A \mid \vec{b})$. Apply reduction

$$E_1 (A \mid \vec{b}) = (E_1 A \mid E_1 \vec{b})$$

$$\vdots$$

$$(E_k \cdots E_2 E_1 A \mid E_k \cdots E_2 E_1 \vec{b})$$

$$(R \mid \vec{c})$$

Read the solution.

e.g. $$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Mult on left by $E_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$:

down elim.

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

up elim.

Mult on left by $E_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{STOP}$$

$$\begin{cases} x - z = 2 \\ y + 2z = -1 \end{cases}$$

Solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2+z \\ -1-2z \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

one particular solution

null space.

Next time:

What if $\vec{b} \notin C(A)$?

$A\vec{x} = \vec{b}$ has no solution.

Maybe look for $\vec{x}$ so distance

$$\|A\vec{x} - \vec{b}\| \text{ is minimized.}$$

Spoiler: $\vec{x} = A(A^TA)^{-1}A^T\vec{b}$

when $(A^TA)^{-1}$ exists.