HW 3 due Mon Oct 17.

Recall: Let A be $m \times n$ matrix.

For any $m \times m$ matrix E, every row of EA is a l.c. of rows of A, hence

$$R(EA) \subseteq R(A).$$

If E^{-1} exists (only need a left inverse) then

$$R(A) = R(E^{-1}EA) \subseteq R(EA).$$

Hence $R(A) = R(EA)$.

For any matrix F, each col AF is a l.c. of cols of A, so

$$C(AF) \subseteq C(A)$$

& if F^{-1} exist (only need a right inverse) then

$$C(AF) = C(A).$$

If E, F invertible, also have isomorphisms

$$R(AF) \cong R(A)$$
$$C(EA) \cong C(A)$$

Key:

$$\dim R(EAF) = \dim R(A)$$
$$\dim C(EAF) = \dim C(A).$$

Fund. Theorem. :

Perform row & col ops. to get

$$EAF = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

$\Longrightarrow \quad \dim R(A) = \dim C(A) = r$

called the rank of A.

Existence of Inverses.

Let A be $m \times n$.

$$N(A) = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \}$$

Note:

$A\vec{x} = \vec{0} \iff \vec{x} \perp$ to every row of A

$$\begin{pmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_m^T \end{pmatrix} \vec{x} = \begin{pmatrix} \vec{a}_1^T \vec{x} \\ \vdots \\ \vec{a}_m^T \vec{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\iff \vec{a}_i^T \vec{x} = 0 \quad \forall i.$$

Jargon: $N(A) = R(A)^{\perp}$

Abstract: Let V be any inner prod. vector space, $\mathcal{U}$ any subspace.

Define the orthogonal complement

$$\mathcal{U}^{\perp} = \{ \vec{v} \in V : \langle \vec{u}, \vec{v} \rangle = 0 \text{ for all } \vec{u} \in \mathcal{U} \}$$

"$\mathcal{U}$ perp"

e.g. $\mathcal{U}$ plane in $\mathbb{R}^3$

$\mathcal{U}^{\perp}$ is the normal line.

HW Exercise:

- $\mathcal{U}^{\perp} \subseteq V$ is also a subspace.
- $\mathcal{U} \cap \mathcal{U}^{\perp} = \{ \vec{0} \}$
- If $\mathcal{U}$ is finite dimensional

then $\forall \vec{v} \in V \; \exists \, \vec{v}_1 \in \mathcal{U}, \vec{v}_2 \in \mathcal{U}^{\perp}$,

$$\vec{v} = \vec{v}_1 + \vec{v}_2 .$$

Jargon: $\mathcal{U} + \mathcal{U}^\perp = V$.

Combining (b,c): $V = \mathcal{U} \oplus \mathcal{U}^\perp$

direct sum.

unique

i.e. $\forall \vec{v} \in V$, $\exists !\ \vec{v}_1 \in \mathcal{U}$, $\vec{v}_2 \in \mathcal{U}^\perp$

$$\vec{v} = \vec{v}_1 + \vec{v}_2$$

[Jargon: Given subspaces $\mathcal{U}, V \leq W$ define new subspaces

$$\mathcal{U} \cap V \ \& \ \mathcal{U} + V = \{\vec{u} + \vec{v}\}.$$]

- If V is finite dim'l then

$$\dim \mathcal{U} + \dim \mathcal{U}^\perp = \dim V.$$

e.g. 2D plane & 1D perp line in $\mathbb{R}^3$

[Generally:

$$\dim \mathcal{U} + \dim V = \dim(\mathcal{U}+V) - \dim(\mathcal{U} \cap V).$$]

Back to matrices : A $m \times n$

$$N(A) \subseteq \mathbb{R}^n$$
$$R(A) \subseteq \mathbb{R}^n$$

And $N(A) = R(A)^{\perp}$

Hence

$$\dim R(A) + \dim N(A) = \overset{\#\text{ cols } A}{n}$$

" Rank-Nullity Theorem "

i.e. $\dim N(A) = \#\text{cols } A - \#\text{ ind rows } A$.

[Independent of Fund Theorem.]

Application [HW] :

New proof of existence of dim.

Ind. $\{\vec{u}_1, \ldots, \vec{u}_k\} \subseteq \mathbb{R}^n$

Span. $\{\vec{v}_1, \ldots, \vec{v}_\ell\} \subseteq \mathbb{R}^n$

Claim: $k \le l$.

Proof: Let $U = (\vec{u}_1 \cdots \vec{u}_k)$ $n \times k$

$V = (\vec{v}_1 \cdots \vec{v}_l)$ $n \times l$.

Since $\vec{v}$ span $\exists A$ $l \times k$.

$$\underset{n\times k}{U} = \underset{n\times l}{V}\,\underset{l\times k}{A}$$

Assume for contradiction that $k > l$, so A is short & wide.

Use Rank-Nullity to show

$$\dim N(A) \geq 1.$$

i.e. $\exists \vec{x} \neq \vec{0}$, $A\vec{x} = \vec{0}$.

But then

$$U\vec{x} = VA\vec{x} = V\vec{0} = \vec{0}$$

contradicts the independence of the $\vec{u}$ vectors.

✓

Existence of Inverse Matrices:

Let A be $m \times n$.

Let $r_R = \dim R(A)$

$r_C = \dim C(A)$

So $\dim N(A) = n - r_R$

$\dim N(A^T) = m - r_C$.

Fund Theorem: $r_R = r_C$.

(1) A has a right inverse

$\iff AX = I_m$

$\iff A\vec{x}_j = \vec{e}_j$ has solution $\vec{x}_j$ for each j.

$\iff \vec{e}_1, \dots, \vec{e}_m \in C(A)$.

$\iff C(A) = \mathbb{R}^m$

$\iff r_C = m$.

(2) similarly, A has a left inverse $\iff r_R = n$.

(3) Combining (1) & (2):

Since $r_R = r_C = r$, see

A has 2-sided inverse

$\iff \quad r = m \;\; \& \;\; r = n.$

(hence $m = n$).

In fact, for square $\overset{n\times n}{A}, B$:

$$AB = I \iff BA = I.$$

Proof:

$AB = I \implies A$ has right inverse

$\implies r = n$

$\implies A$ has a left inverse

Say $CA = I$.

But then

$$B = IB = CAB = CI = C$$

so $B = C$ is a 2-sided inverse.

Hence $BA = I$.

Remark: This proof is surprisingly indirect!