HW 4 Discussion:

Problem 1: $k > n \Rightarrow \dim A^k(\mathbb{R}^n) = 0$.

(a) A has repeated column.

Say $A' = A$ where A' obtained from A by switching 2 columns.

Then from def. of "alternating"

$$\varphi(A') = -\varphi(A).$$

O.T.O.H, $A' = A \Rightarrow \varphi(A') = \varphi(A)$.

So
$$\varphi(A) = -\varphi(A)$$
$$2\varphi(A) = 0$$
$$\varphi(A) = 0.$$

(b) A has dependent cols.

e.g. $A = \left(\vec{a}_1 \,\middle|\, \vec{a}_2 \,\middle|\, \lambda \vec{a}_1 + \mu \vec{a}_2 \right)$.

By "multilinearity":

$$\varphi(A) = \lambda \cdot \cancel{\varphi\left(\vec{a}_1 \,\middle|\, \vec{a}_2 \,\middle|\, \vec{a}_1 \right)}^{\,0} + \mu \cdot \cancel{\varphi\left(\vec{a}_1 \,\middle|\, \vec{a}_2 \,\middle|\, \vec{a}_2 \right)}^{\,0} = 0.$$

(c) Let $k > n$, $\varphi \in \mathcal{A}^k(\mathbb{R}^n)$.

Any set of k vectors in $\mathbb{R}^n$ must be dependent. So, for any $\vec{a}_1, \dots, \vec{a}_k \in \mathbb{R}^n$ have

$$\varphi(\vec{a}_1, \dots, \vec{a}_k) = 0.$$

In other words $\varphi = 0$, i.e., the zero function $(\mathbb{R}^n)^k \to \mathbb{R}$.

Remark: For $0 \le k \le n$ I mentioned that $\dim \mathcal{A}^k(\mathbb{R}^n) = \binom{n}{k}$.

[Convention $\mathcal{A}^0 = \mathbb{R}$, $\mathcal{A}^1 = \mathcal{T}^1$,

so $\dim \mathcal{A}^0 = 1 = \binom{n}{0}$

$\dim \mathcal{A}^1 = n = \binom{n}{1}$ ✓]

Proved in class

$$\mathcal{A}^n(\mathbb{R}^n) = \operatorname{span}\{\det\}.$$

More generally, given $n \times k$ matrix ($0 \le k \le n$) and subset $I \subseteq \{1, \dots, n\}$, define the function $\det_I : A^k(\mathbb{R}^n)$

by $\det_I(A) =$ determinant of $k \times k$ submatrix of A consisting of rows I.

e.g. $\det_{\{2,3\}} \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} = 1 \cdot 3 - 2 \cdot 2 = -1.$

<u>Theorem</u>: For any $0 \le k \le n$, the functions $\det_I$ with $I \subseteq \{1, \dots, n\}$ of size k are a <u>basis</u> for $A^k(\mathbb{R}^n)$.

Hence

$$\dim A^k(\mathbb{R}^n) = \# \text{ subsets of } \{1, \dots, n\} \text{ of size } k$$

$$= \binom{n}{k}.$$

Problem 2: Volume.

e.g. $k=3$. Consider $\vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathbb{R}^n$.

$$A = \left(\vec{a}_1 \,\middle|\, \vec{a}_2 \,\middle|\, \vec{a}_3 \right) \quad n \times 3.$$

3×3

$$A^T A = \begin{pmatrix} \vec{a}_1^{\,T} \\ \hline \vec{a}_2^{\,T} \\ \hline \vec{a}_3^{\,T} \end{pmatrix} \left(\vec{a}_1 \,\middle|\, \vec{a}_2 \,\middle|\, \vec{a}_3 \right)$$

$$= \begin{pmatrix} \|\vec{a}_1\|^2 & \vec{a}_1 \cdot \vec{a}_2 & \vec{a}_1 \cdot \vec{a}_3 \\ \vec{a}_1 \cdot \vec{a}_2 & \|\vec{a}_2\|^2 & \vec{a}_2 \cdot \vec{a}_3 \\ \vec{a}_1 \cdot \vec{a}_3 & \vec{a}_2 \cdot \vec{a}_3 & \|\vec{a}_3\|^2 \end{pmatrix}$$

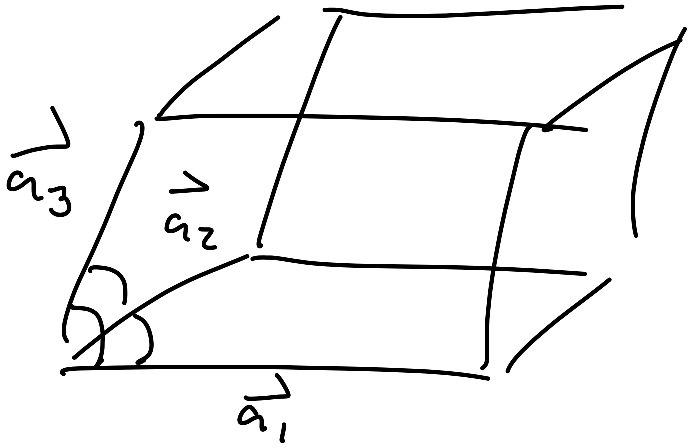
Use $\vec{a}_i \cdot \vec{a}_j = \|\vec{a}_i\| \, \|\vec{a}_j\| \cos\theta_{ij}$

Plug into $A^T A$ to get

$$\det(A^T A) = \|\vec{a}_1\|^2 \|\vec{a}_2\|^2 \|\vec{a}_3\|^2 \cdot$$

$$\Big(1 + 2\cos\theta_{12}\cos\theta_{13}\cos\theta_{23} - \left(\cos^2\theta_{12} + \cos^2\theta_{13} + \cos^2\theta_{23}\right)\Big)$$

$=$ Volume^2 of parallelepiped (3D) gen. by $\vec{a}_1, \vec{a}_2, \vec{a}_3$.



Since the formula depends only on lengths & angles, it holds in any # of dimensions.

$$\mathrm{Vol}_k(A) = \sqrt{\det(A^TA)}$$

True even when A not square.

Special case $k=n$,

$$\mathrm{Vol}_n(A) = \sqrt{\det(A^TA)}$$

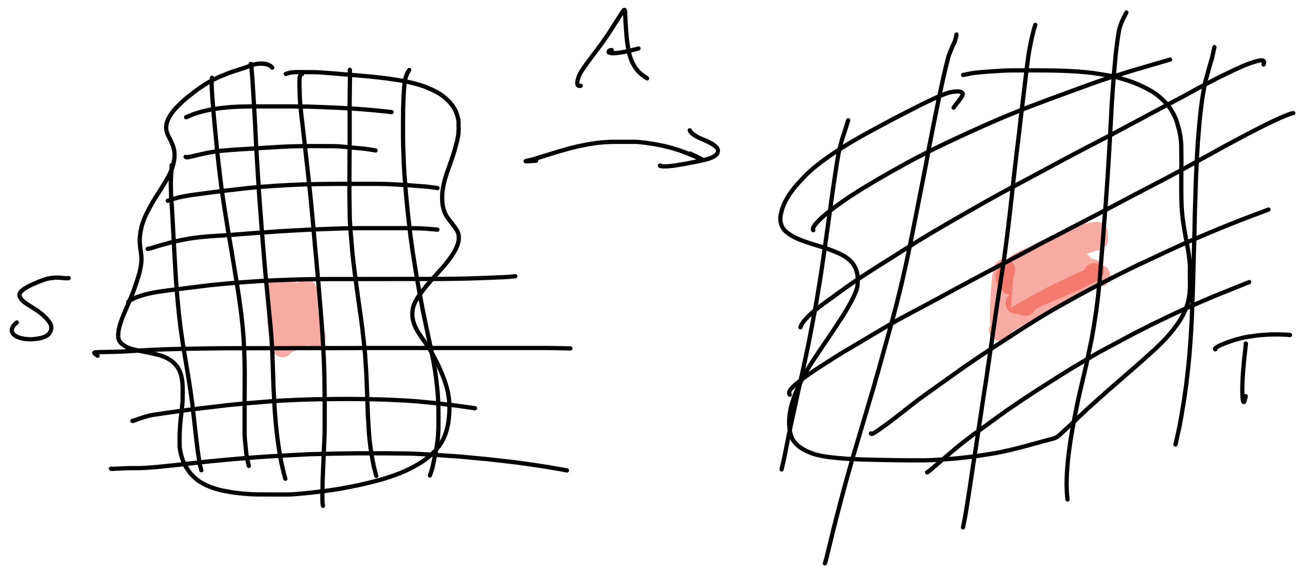
$$= \sqrt{\det(A)\det(A)}$$

$$= \sqrt{\det(A)^2} = |\det(A)|.$$

only makes sense for SQUARE A.

Application To Calculus.

Linear $A : \mathbb{R}^n \to \mathbb{R}^m$



say A maps set $S \subseteq \mathbb{R}^n$ onto set $T \subseteq \mathbb{R}^m$. Then

$$\mathrm{Vol}_n(T) = \sqrt{\det(A^TA)} \cdot \mathrm{Vol}_n(S)$$

Non-linear $\vec{r} : \mathbb{R}^n \to \mathbb{R}^m$.

$$\vec{r}(\vec{p}) = (r_1(\vec{p}), \ldots, r_m(\vec{p}))$$

where each $r_i : \mathbb{R}^n \to \mathbb{R}$

Taylor expand each component $r_i : \mathbb{R}^n \to \mathbb{R}$ at $\vec{p} \in \mathbb{R}^n$:

$$r_i(\vec{p}+\vec{x}) = r_i(\vec{p}) + (\nabla r_i)^T_{\vec{p}}\, \vec{x} + h.t.$$

Then

$$\vec{r}(\vec{p}+\vec{x}) = \vec{r}(\vec{p}) + \underbrace{\begin{pmatrix} (\nabla r_1)^T_{\vec{p}} \\ \vdots \\ (\nabla r_m)^T_{\vec{p}} \end{pmatrix}}_{m \times n \text{ matrix}} \vec{x} + \cdots$$

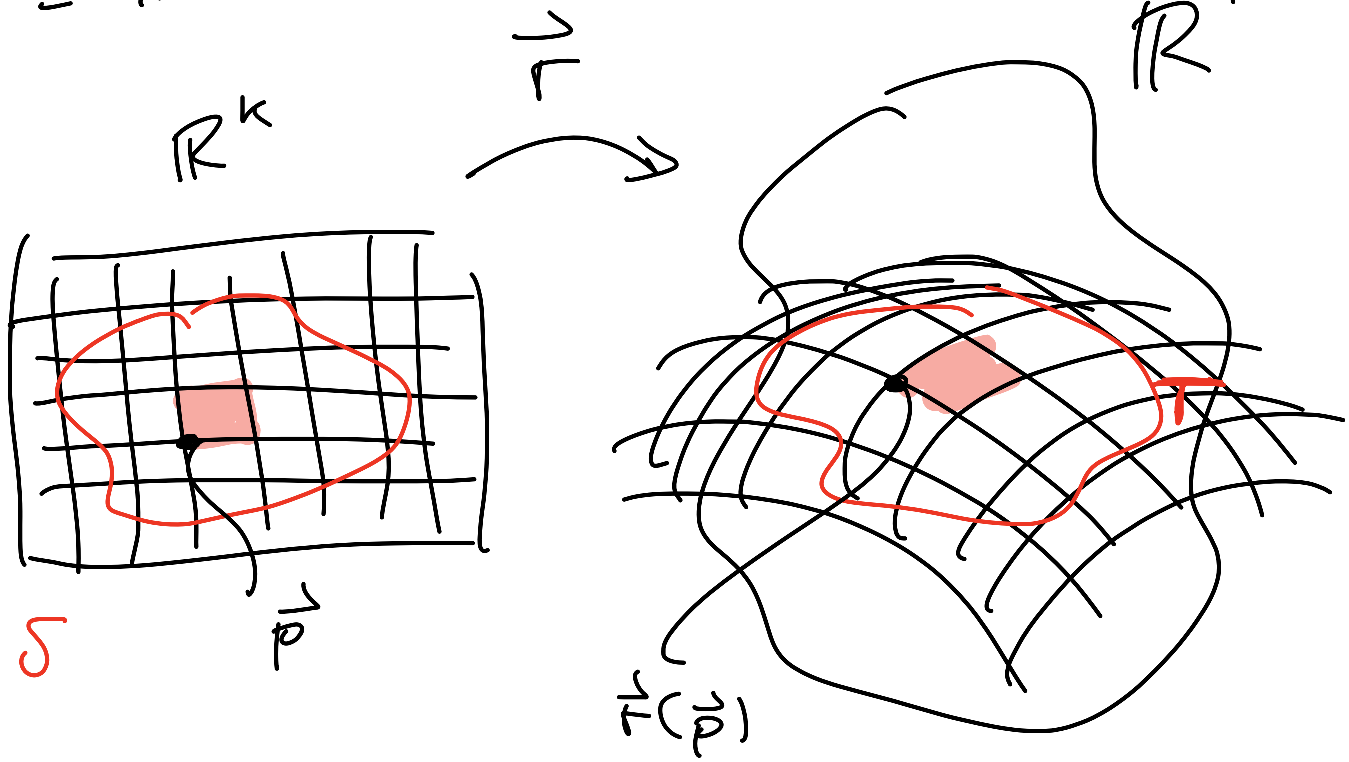
Jacobian Matrix:

$$J\vec{r} = \begin{pmatrix} \nabla r_1^T \\ \vdots \\ \nabla r_m^T \end{pmatrix} = \begin{pmatrix} \partial r_1/\partial x_1 & \cdots & \partial r_1/\partial x_n \\ \vdots & & \\ \partial r_m/\partial x_1 & \cdots & \partial r_m/\partial x_n \end{pmatrix}$$

$$\underset{m\times 1}{\vec{r}(\vec{p}+\vec{x})} = \underset{m\times 1}{\vec{r}(\vec{p})} + \underbrace{\underset{m\times n}{(J\vec{r})_{\vec{p}}}\ \underset{n\times 1}{\vec{x}}}_{m\times 1} + \cdots$$

Near $\vec{p}$ the volume gets stretched by Factor of $\sqrt{\det((J\vec{r})^T(J\vec{r}))}$

THINK: $\vec{r}: \mathbb{R}^k \to \mathbb{R}^n$ as a "parametrization" of a region $T \subseteq \mathbb{R}^n$



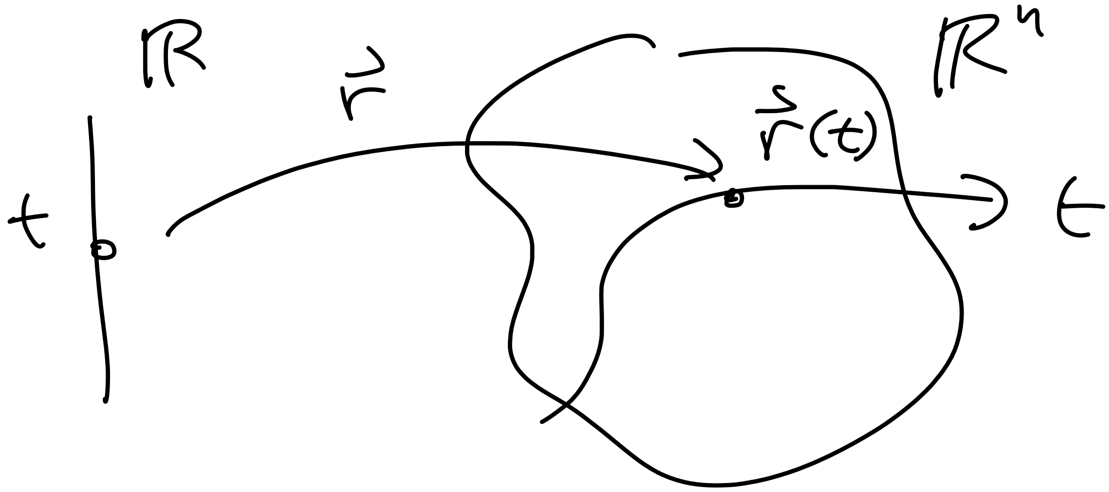
Near $\vec{p}$, $\vec{r}$ stretches volume by

$$\sqrt{\det((J\vec{r})_{\vec{p}}^T(J\vec{r}))}$$

To compute k-volume of k-region $T \subseteq \mathbb{R}^n$ we integrate all the little pieces of k-volume

$$\mathrm{Vol}_k(T) = \underbrace{\int_{\vec{p}\in S} \sqrt{\det\left((J\vec{r})_{\vec{p}}^T (J\vec{r})_{\vec{p}}\right)}\, d\vec{p}}_{\text{regular multiple integral in Euclidean } \mathbb{R}^k}$$

e.g. Parametrized Path in $\mathbb{R}^n$



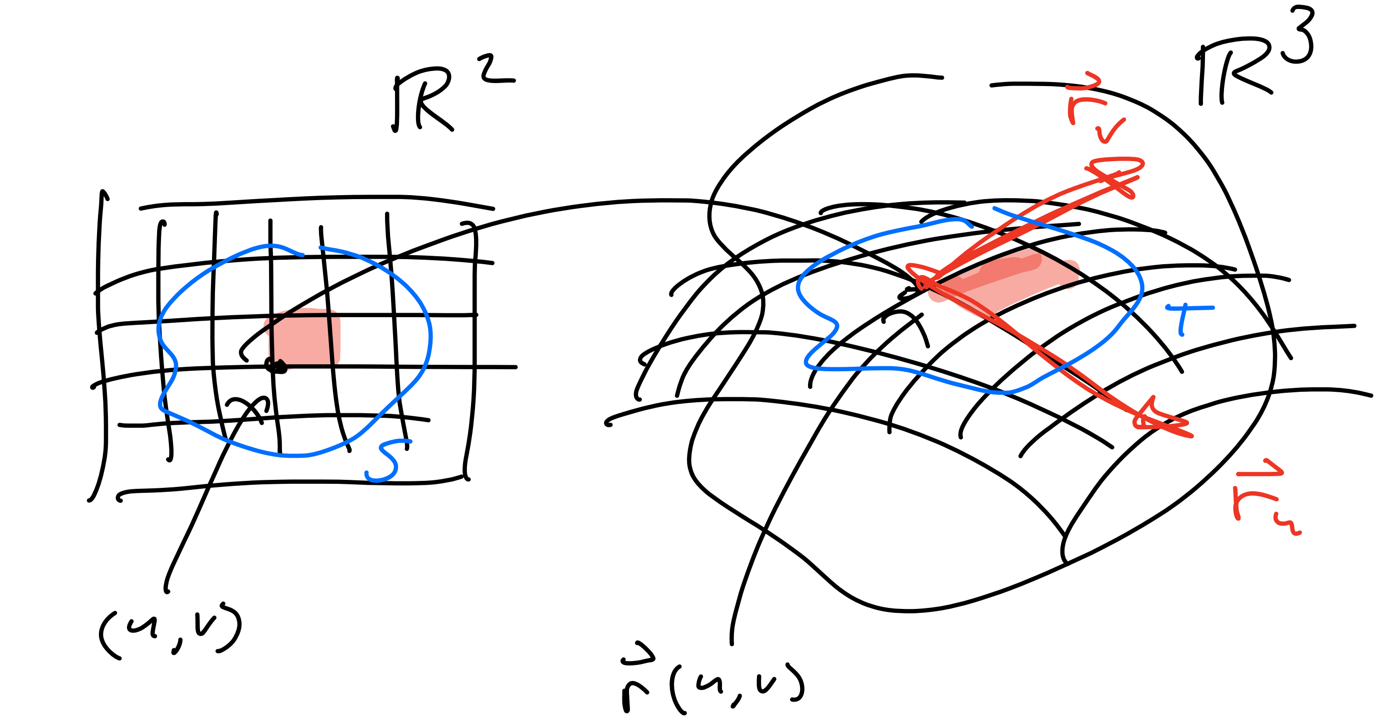
$$J\vec{r} = \nabla\vec{r} = \begin{pmatrix} \partial r_1/\partial t \\ \partial r_2/\partial t \\ \vdots \\ \partial r_n/\partial t \end{pmatrix} = \vec{r}\,'(t)$$

velocity.

$$\text{Arc Length} = \int \sqrt{\vec{r}\,'(t)^T \vec{r}\,'(t)}\, dt$$

$$= \int \|\vec{r}\,'(t)\|\, dt$$

Parametrized Surface in $\mathbb{R}^3$.



$$J\vec{r} = \begin{pmatrix} \partial r_1/\partial u & \partial r_1/\partial v \\ \partial r_2/\partial u & \partial r_2/\partial v \\ \partial r_3/\partial u & \partial r_3/\partial v \end{pmatrix} = \left(\vec{r}_u \,\middle|\, \vec{r}_v \right)$$

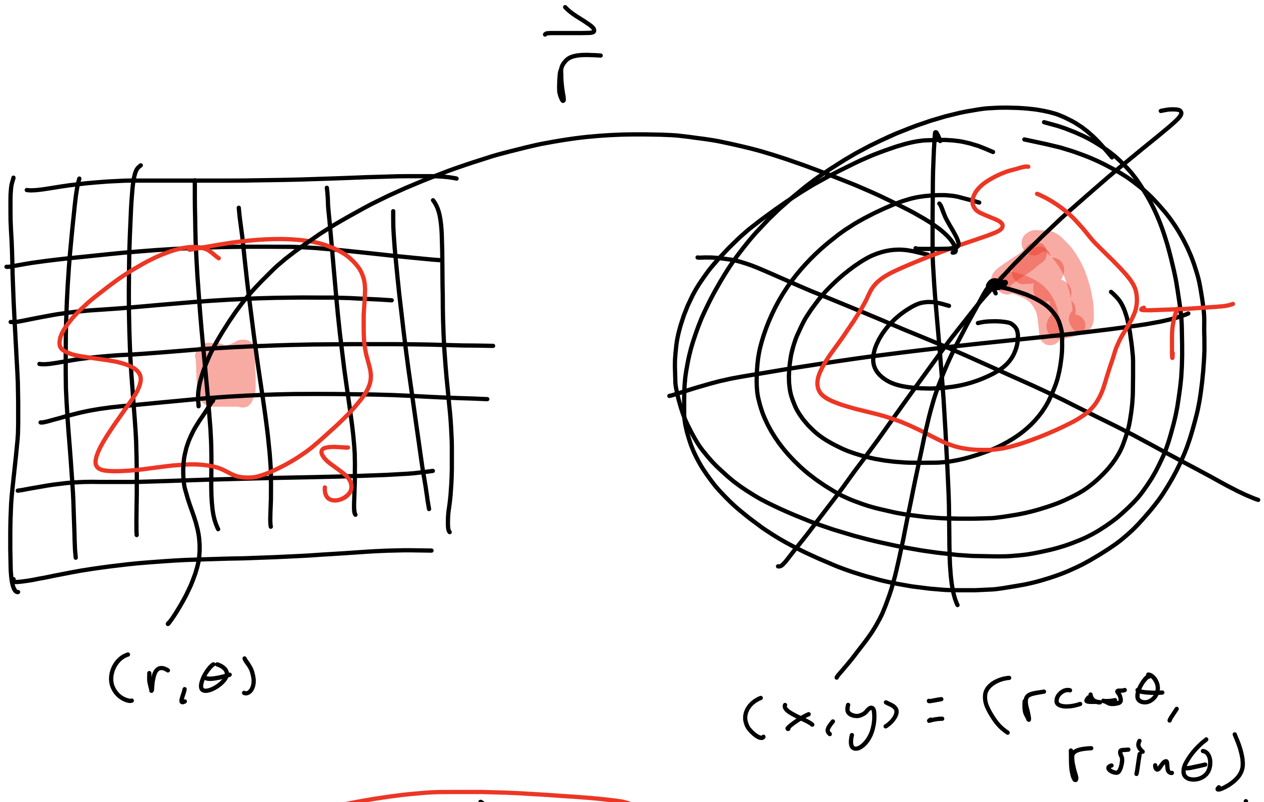
$$\sqrt{\det((J\vec{r})^T(J\vec{r}))} = \| \underbrace{\vec{r}_u \times \vec{r}_v}_{\text{peculiar 3D construction.}} \|$$

$$\text{Surface Area} = \int_S \|\vec{r}_u \times \vec{r}_v\|(u,v)\, du\, dv$$

Change of Coordinates:

$$\vec{r} : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$



$$J\vec{r} = \begin{pmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{pmatrix}$$

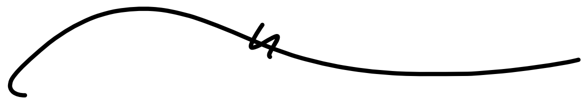
gradient vectors.

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\sqrt{\det\left((J\vec{r})^T (J\vec{r})\right)} = \left| \det(J\vec{r}) \right|$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

$$\text{Area of } T = \int_{(r,\theta)\in S} r \, dr \, d\theta .$$



All the same!

Big Idea:

Alternating k-forms = Integrands