HW due Mon Oct 31.

Proposal: Move Exam 2 to Fri Nov 4.

Current Topic: Determinants.

Recall: linear
bilinear
multilinear forms

$$\varphi : V^k \to \mathbb{R}$$

linear in each position.

Warning:

$$\text{multilinear} \neq \text{linear}.$$

e.g. 2-form $\varphi_B(\vec{x}, \vec{y}) = \vec{x}^T B \vec{y}$

$$\varphi_B(\text{matrix } X) = \varphi_B\left(\begin{matrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{matrix}\right)$$

$$= \vec{x}_1^T B \vec{x}_2$$

$$\varphi_B(X + Y) \neq \varphi_B(X) + \varphi_B(Y)$$

$$\varphi_B(X) = \vec{x}_1^{\,T} B \vec{x}_2$$
$$\varphi_B(Y) = \vec{y}_1^{\,T} B \vec{y}_2$$
$$\varphi_B(X+Y) = (\vec{x}_1+\vec{y}_1)^T B (\vec{x}_2+\vec{y}_2)$$
$$= \varphi_B(X) + \varphi_B(Y)$$
$$\vec{x}_1^{\,T} B \vec{y}_2 + \vec{y}_1^{\,T} B \vec{x}_2 \neq 0.$$

[Relevant: $\det(A+B) \neq \det(A) + \det(B)$.]

$S^k(\mathbb{R}^n)$ = symmetric k-forms

$A^k(\mathbb{R}^n)$ = alternating k-forms

To save space, write

$$\varepsilon_{ij} = \varepsilon_i \otimes \varepsilon_j \,, \quad \varepsilon_{ijk} = \varepsilon_i \otimes \varepsilon_j \otimes \varepsilon_k$$

"Standard basis" of $\mathcal{T}^k(\mathbb{R}^n)$

is $\varepsilon_{i_1 i_2 \cdots i_k}$, $i_1, \dots, i_k \in \{1, \dots, n\}$

$\dim \mathcal{T}^k(\mathbb{R}^n) = n^k$.

Examples of $\mathcal{S}^k$ & $\mathcal{A}^k$:

$\mathcal{S}^1(\mathbb{R}^2) = \text{span}\{\varepsilon_1, \varepsilon_2\}$ last time

$\mathcal{S}^2(\mathbb{R}^2) = \text{span}\{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12} + \varepsilon_{21}\}$

$$(\varepsilon_{12} + \varepsilon_{21})\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1 b_2 + a_2 b_1$$

switching $a \leftrightarrow b$ does nothing:

$b_1 a_2 + b_2 a_1 = a_1 b_2 + a_2 b_1$ ✓

So $\dim \mathcal{S}^2(\mathbb{R}^2) = 3$.

Compare to $\dim \mathcal{T}^2(\mathbb{R}^2) = 4$.

$\mathcal{T}^2 = \text{span}\{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \varepsilon_{21}\}$

$\mathcal{S}^2 = \text{span}\{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12} + \varepsilon_{21}\}$

One more: $\mathcal{S}^3(\mathbb{R}^2)$

$$= \text{span} \{ \varepsilon_{111}, \varepsilon_{222}, \varepsilon_{333},$$

$$\varepsilon_{112} + \varepsilon_{121} + \varepsilon_{211},$$

$$\varepsilon_{122} + \varepsilon_{212} + \varepsilon_{221},$$

$\vdots$ 3 skipped.

$$\varepsilon_{233} + \varepsilon_{323} + \varepsilon_{332},$$

$$\varepsilon_{123} + \varepsilon_{213} + \varepsilon_{213} + \varepsilon_{231} + \varepsilon_{312} + \varepsilon_{321} \}$$

$$\dim S^3(\mathbb{R}^2) = 10.$$

General:

$$\dim S^k(\mathbb{R}^n) = \binom{n+k-1}{k}.$$

Examples of $A^k(\mathbb{R}^n)$:

$$A^1(\mathbb{R}^2) = \text{span}\{\varepsilon_1, \varepsilon_2\}$$

$$A^2(\mathbb{R}^2) = \text{span}\{\varepsilon_{12} - \varepsilon_{21}\}$$

$$\dim A^2(\mathbb{R}^2) = 1.$$

$$(\varepsilon_{12} - \varepsilon_{21})\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1 b_2 - a_2 b_1$$

This is the 2×2 determinant!

Alternating: switch $a \leftrightarrow b$,

$$b_1 a_2 - b_2 a_1 = -(a_1 b_2 - a_2 b_1)$$

$A^3(\mathbb{R}^2) = \{0\}$ nothing.

$A^k(\mathbb{R}^2) = \{0\}$ $k \geq 3$.

[HW 4.1]

$A^1(\mathbb{R}^3) = \operatorname{span}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$

$A^2(\mathbb{R}^3) = \operatorname{span}\{\varepsilon_{12} - \varepsilon_{21}, \varepsilon_{13} - \varepsilon_{31}, \varepsilon_{23} - \varepsilon_{32}\}$.

$A^3(\mathbb{R}^3) =$

$$\operatorname{span}\{\underbrace{\varepsilon_{123} + \varepsilon_{231} + \varepsilon_{312} - \varepsilon_{132} - \varepsilon_{213} - \varepsilon_{321}}_{\delta}\}$$

So $\dim A^3(\mathbb{R}^3) = 1$.

The unique alternating 3-form on $\mathbb{R}^3$ is the determinant:

$$\delta \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2$$
$$- a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1$$

This is multilinear in the 3-columns & alternating (switch a, b, c in any way, see what happens ...)

General:

$$\dim \mathcal{A}^k(\mathbb{R}^n)$$

$$= \begin{cases} 0 & k > n \\ \binom{n}{k} & 0 \le k \le n. \end{cases}$$

[Convention: $\mathcal{A}^0(\mathbb{R}^n) = \{0\}$.]

Today: Proof that

$$\dim A^n(\mathbb{R}^n) \leq 1.$$

Proof: Suppose $\delta \in A^n(\mathbb{R}^n)$.

$\delta(A) = \delta(\vec{a}_1, \dots, \vec{a}_n)$. A $n \times n$ matrix.

- Alternating in Columns
- Multilinear in Columns
- Normalize $\delta(I_n) = 1$.

$$\delta(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n) = 1.$$

Show if δ_1 & δ_2 satisfy these properties then $\delta_1 = \delta_2$.

<u>Lemma 1:</u>

IF A has a repeated column then $\delta(A) = 0$.

Indeed suppose $\vec{a}_i = \vec{a}_j$.

Let $A' = A$ with cols i & j switched.

On the one hand $A' = A$.

On the other hand:

Alternating $\Rightarrow$ $\delta(A') = -\delta(A)$.

But then

$$\delta(A) = -\delta(A')$$

$$\delta(A) = -\delta(A)$$

$$2\delta(A) = 0$$

$$\delta(A) = 0$$

Follows: A has depend. cols $\Rightarrow$ $\delta(A) = 0$.

Lemma 2: Elementary Matrices.

$$D_i(\lambda) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$L_{ij}(\lambda) = \begin{array}{c c} & \begin{matrix} & & j & \end{matrix} \\ i & \begin{pmatrix} 1 & \cdots & \lambda & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \end{array}$$

$$T_{ij} = \begin{array}{c c} & \begin{matrix} & i & & j & \end{matrix} \\ \begin{matrix} \\ i \\ \\ j \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & & \\ & 0 & \cdots & 1 & \\ & \vdots & & \vdots & \\ & 1 & \cdots & 0 & \\ & & & & 1 \end{pmatrix} \end{array}$$

Then $\delta(A\,D_i(\lambda)) = \lambda\,\delta(A)$

$\delta(A\,L_{ij}(\lambda)) = \delta(A)$

$\delta(A\,T_{ij}) = -\delta(A)$

Omit the proof (see notes).

Taking $A = I$:

$$\delta(D_i(\lambda)) = \lambda$$

$$\delta(L_{ij}(\lambda)) = 1.$$

$$\delta(T_{ij}) = -1.$$

Proof: If A^{-1} does not exist then A has dep cols so $\delta(A) = 0$.

Suppose A^{-1} exists. Reduce

$$A E_1 E_2 \cdots E_k = I$$

$$A = E_k^{-1} \cdots E_2^{-1} E_1^{-1}$$

$$\delta(A) = \delta(E_k^{-1}) \cdots \delta(E_1^{-1}).$$

If δ_1, δ_2 both satisfy rules

$$\Rightarrow \delta_1(A) = \delta_1(E_k^{-1}) \cdots \delta_1(E_1^{-1})$$

$$\|$$

$$\delta_2(A) = \delta_2(E_k^{-1}) \cdots \delta_2(E_1^{-1})$$

Q.E.D.

Use the same Lemma 2 to prove

$$\delta(A^T) = \delta(A)$$

$$\delta(AB) = \delta(A)\delta(B).$$

Proof: Assume A^{-1} exists so $\delta(A) \neq 0$. We can write.

Then the result is easy.

$$A = E_1 E_2 \cdots E_k.$$

$$A^T = E_k^T \cdots E_2^T E_1^T$$

But $\delta(E^T) = \delta(E)$.

so $\delta(A) = \delta(E_1) \cdots \delta(E_k)$

$$= \delta(E_k) \cdots \delta(E_1)$$

$$= \delta(E_k^T) \cdots \delta(E_1^T)$$

$$= \delta(A^T).$$

Also let $B = \underbrace{F_1 \cdots F_\ell}_{\text{elementary}}$.

$$\delta(AB) = \delta(E_1 \cdots E_k F_1 \cdots F_\ell)$$

$$= \delta(E_1) \cdots \delta(E_k)\, \delta(F_1) \cdots \delta(F_\ell)$$

$$= \delta(A)\, \delta(B) \quad \checkmark$$

So far we did not prove

$$\dim A^n(\mathbb{R}^n) \neq 0.$$