This Week: Determinants.

Linear Forms ✓
Bilinear Forms ✓

A multilinear k-form on $V/\mathbb{R}$ is a function

$$\varphi : \underbrace{V \times V \times \cdots \times V}_{V^k} \to \mathbb{R}$$

satisfying

$$\varphi\left(\sum a_i \vec{u}_i, \vec{v}_2, \ldots, \vec{v}_k \right)$$

$$= \sum a_i \varphi(\vec{u}_i, \vec{v}_2, \ldots, \vec{v}_k)$$

etc.

Let $\mathcal{T}^k(V)$ be the set of k-forms on V, which is itself a vector space:

$$(\varphi + a\psi)(\vec{v}_1, \ldots, \vec{v}_k)$$
$$= \varphi(\vec{v}_1, \ldots, \vec{v}_k) + a\psi(\vec{v}_1, \ldots, \vec{v}_k)$$

$\mathcal{T}^1(V)$ = 1-forms = linear forms.

$\mathcal{T}^1(\mathbb{R}^n) \leftrightarrow$ row vectors

$$\dim \mathcal{T}^1(\mathbb{R}^n) = n.$$

$\mathcal{T}^2(V)$ = 2-forms = bilinear forms

$\mathcal{T}^2(\mathbb{R}^n) \leftrightarrow$ $n \times n$ matrices

$$\dim \mathcal{T}^2(\mathbb{R}^n) = n^2$$

In general, I claim that

$$\dim \mathcal{T}^k(\mathbb{R}^n) = n^k.$$

HOW? We need a basis.

Let $\vec{e}_1, \ldots, \vec{e}_n \in \mathbb{R}^n$ be the standard basis. Define a "standard basis" $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in (\mathbb{R}^n)^\vee = \mathcal{T}^1(\mathbb{R}^n)$.

$$\varepsilon_i : \mathbb{R}^n \to \mathbb{R}$$
$$\vec{x} \mapsto \varepsilon_i(\vec{x})$$

Define by $\varepsilon_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i$

Then $\varepsilon_1, \dots, \varepsilon_n$ is the "dual basis" to $\vec{e}_1, \dots, \vec{e}_n$:

$$\varepsilon_i(\vec{e}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Can think

$$\vec{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \qquad \varepsilon_i = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}$$

Define tensor product of 1-forms:

For any i, j $\varepsilon_i \otimes \varepsilon_j$ is a 2-form:

$$\varepsilon_i \otimes \varepsilon_j (\vec{x}_1, \vec{x}_2) = \varepsilon_i(\vec{x}_1)\, \varepsilon_j(\vec{x}_2)$$

e.g. $\varepsilon_1 \otimes \varepsilon_2 \left(\begin{matrix} 1 \\ -1 \\ 1 \end{matrix}, \begin{matrix} 2 \\ 3 \\ 4 \end{matrix} \right) = \varepsilon_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \varepsilon_2 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$= 1 \cdot 3 = 3.$

Note that $\varepsilon_1 \otimes \varepsilon_2 \neq \varepsilon_2 \otimes \varepsilon_1$:

$$\varepsilon_2 \otimes \varepsilon_1 \left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right) = (-1) \cdot 2 = -2$$

More generally:

$$\varepsilon_{i_1} \otimes \varepsilon_{i_2} \otimes \cdots \otimes \varepsilon_{i_k} (\vec{x}_1, \ldots, \vec{x}_k)$$

$$= \varepsilon_{i_1}(\vec{x}_1)\, \varepsilon_{i_2}(\vec{x}_2) \cdots \varepsilon_{i_k}(\vec{x}_k)$$

<u>Theorem</u>: $\mathcal{T}^k(\mathbb{R}^n)$ has basis

$$\varepsilon_{i_1} \otimes \varepsilon_{i_2} \otimes \cdots \otimes \varepsilon_{i_k}$$

n^k of these

for all $i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$.

Hence $\dim \mathcal{T}^k(\mathbb{R}^n) = n^k$.

e.g.

$$\mathcal{T}^1(\mathbb{R}^n) = \operatorname{span}\{\varepsilon_1, \dots, \varepsilon_n\}$$
$$= \left\{ \sum b_i \varepsilon_i \right\}$$
$$\longleftrightarrow \left\{ \text{row vectors } (b_1 \cdots b_n) \right\}$$

$$\mathcal{T}^2(\mathbb{R}^n) = \operatorname{span}\{\varepsilon_i \otimes \varepsilon_j\}$$
$$= \left\{ \sum b_{ij} \varepsilon_i \otimes \varepsilon_j \right\}$$
$$\longleftrightarrow \left\{ \text{matrices } \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \right\}$$

$$\mathcal{T}^3(\mathbb{R}^n) = \operatorname{span}\{\varepsilon_i \otimes \varepsilon_j \otimes \varepsilon_k\}$$
$$= \left\{ \sum b_{ijk} \varepsilon_i \otimes \varepsilon_j \otimes \varepsilon_k \right\}$$
$$\longleftrightarrow \left\{ \text{cubes of numbers } b_{ijk} \right\}$$

"3-tensors"

Two special kinds of tensors:

- symmetric.

$\varphi \in \mathcal{T}^k(V)$ is symmetric if switching any two inputs does change the output.

e.g. $\varepsilon_i \otimes \varepsilon_j + \varepsilon_j \otimes \varepsilon_i$ is symm.

$$(\varepsilon_1 \otimes \varepsilon_2 + \varepsilon_2 \otimes \varepsilon_1)\begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= \varepsilon_1 \otimes \varepsilon_2 \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 1 & 4 \end{pmatrix} + \varepsilon_2 \otimes \varepsilon_1 \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= 1 \cdot 3 + (-1) \cdot 2 = 1.$$

Switching inputs:

$$(\varepsilon_1 \otimes \varepsilon_2 + \varepsilon_2 \otimes \varepsilon_1)\begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 4 & 1 \end{pmatrix}$$

$$= \varepsilon_1 \otimes \varepsilon_2 \begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 4 & 1 \end{pmatrix} + \varepsilon_2 \otimes \varepsilon_1 \begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 4 & 1 \end{pmatrix}$$

$$= 2 \cdot (-1) + 3 \cdot 1 = 1 \quad \checkmark$$

• Antisymmetric.

Say $\varphi \in \mathcal{T}^k(V)$ is anti-symm if switching any two inputs multiplies output by -1.

e.g. $\varepsilon_1 \otimes \varepsilon_2 - \varepsilon_2 \otimes \varepsilon_1$ is antisymm.

$$(\varepsilon_1 \otimes \varepsilon_2 - \varepsilon_2 \otimes \varepsilon_1)\begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 1 & 4 \end{pmatrix} = 1 \cdot 3 - (-1) \cdot 2 = 5$$

$$(\varepsilon_1 \otimes \varepsilon_2 - \varepsilon_2 \otimes \varepsilon_1)\begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 4 & 1 \end{pmatrix} = 2 \cdot (-1) - 3 \cdot 1 = -5 \quad \checkmark$$

Let $S^k(V)$ = space of symm.

$\Lambda^k(V)$ = space of anti-symm

Then $\dim S^k(\mathbb{R}^n) = \binom{n+k-1}{k}$

$\dim \Lambda^k(\mathbb{R}^n) = \binom{n}{k}$

We will not prove this!

Except for a few special cases.

HW4.1:

$$\dim \Lambda^k(\mathbb{R}^n) = 0 \text{ for } k > n.$$

In Class:

$$\underbrace{\dim \Lambda^n(\mathbb{R}^n) = 1}_{\text{this must be special}}. \quad = \binom{n}{n}$$

The unique (up to scalar mult) anti-symmetric n-form on $\mathbb{R}^n$ is called the "determinant".

e.g.

$$\Lambda^2(\mathbb{R}^2) = \underbrace{\operatorname{span}\{\varepsilon_1 \otimes \varepsilon_2 - \varepsilon_2 \otimes \varepsilon_1\}}_{\text{1-dimensional}}$$

$$(\varepsilon_1 \otimes \varepsilon_2 - \varepsilon_2 \otimes \varepsilon_1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

can also multiply by any scalar

$$\lambda(\varepsilon_1 \otimes \varepsilon_2 - \varepsilon_2 \otimes \varepsilon_1)$$

for any λ. We choose λ so

det(identity matrix) = 1.

$$(\varepsilon_1 \otimes \varepsilon_2 - \varepsilon_2 \otimes \varepsilon_1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1. \checkmark$$

Goal:

- Existence & Uniqueness of the determinant.
- Algebraic Properties.
- How to compute.

key algebraic properties:

$$\det(A^T) = \det(A)$$

$$\det(AB) = \det(A)\det(B)$$

$$\det(A) \neq 0 \iff A^{-1} \text{ exists}$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$