Taylor Expansion:

Differentiable functions can be approximated by polynomials.

Function $f: \mathbb{R}^n \to \mathbb{R}$

$$f(x_1, x_2, \ldots, x_n)$$

Let $f_i = \dfrac{\partial f}{\partial x_i}$

$$f_{ij} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f$$

Assume all $f_i$, $f_{ij}$ exist and are continuous, so that

$$f_{ij} = f_{ji} \quad \text{Clairaut's Theorem.}$$

Write $\vec{x} = (x_1, \ldots, x_n)$

Consider a point $\vec{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$

Define gradient vector

$$\nabla f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$$(\nabla F)_{\vec{p}} = \nabla f \text{ evaluated at } \vec{p}$$

$$= \begin{pmatrix} f_1(\vec{p}) \\ \vdots \\ f_n(\vec{p}) \end{pmatrix}$$

$$Hf = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}$$

$$(Hf)_{\vec{p}} = Hf \text{ evaluated at } \vec{p}.$$

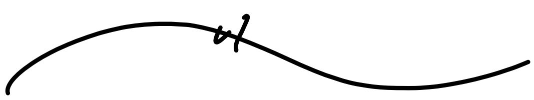
Then:

small

$$f(\vec{p} + \vec{x}) = f(\vec{p})$$

$$+ (\nabla F)_{\vec{p}}^{T} \vec{x} + \frac{1}{2} \vec{x}^{T} (Hf)_{\vec{p}} \vec{x}$$

$$+ \text{ higher terms.}$$



e.g. $f(x,y) = 2 + x - y + 3x^2 + 2xy + 4y^2$

$$f_1 = 1 + 6x + 2y$$

$$f_2 = -1 + 2x + 8y$$

$$f_{11} = 6$$

$$f_{12} = 2$$

$$f_{21} = 2$$

$$f_{22} = 8$$

$$Hf = \begin{pmatrix} 6 & 2 \\ 2 & 8 \end{pmatrix}$$

$$\nabla F = \begin{pmatrix} 1 + 6x + 2y \\ -1 + 2x + 8y \end{pmatrix}$$

$$(\nabla F)_{(0,0)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Expand near $\vec{p} = (0,0)$.

$$f(\vec{p} + \vec{x}) = f((0,0) + (x,y))$$

$$= f(0,0) + (\nabla F)^T_{(0,0)} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$+ \frac{1}{2}(x \ y) (Hf)_{(0,0)} \begin{pmatrix} x \\ y \end{pmatrix} + \cdots$$

$$= 2 + \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$+ \frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 6 & 2 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$+ 0.$$

Expand near $\vec{p} = (1,1)$.

$$f(\vec{p} + \vec{x}) = f((1,1) + (x,y))$$

$$= f(1,1) + (\nabla f)^T_{(1,1)} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$+ \frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \left( Hf \right)_{(1,1)} \begin{pmatrix} x \\ y \end{pmatrix} + 0.$$

$$= 11 + \begin{pmatrix} 9 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} +$$

$$\frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 6 & 2 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 0.$$

Expand near $\underbrace{\vec{p} = \left( \frac{-10}{44}, \frac{8}{44} \right)}_{\text{critical point}}$.

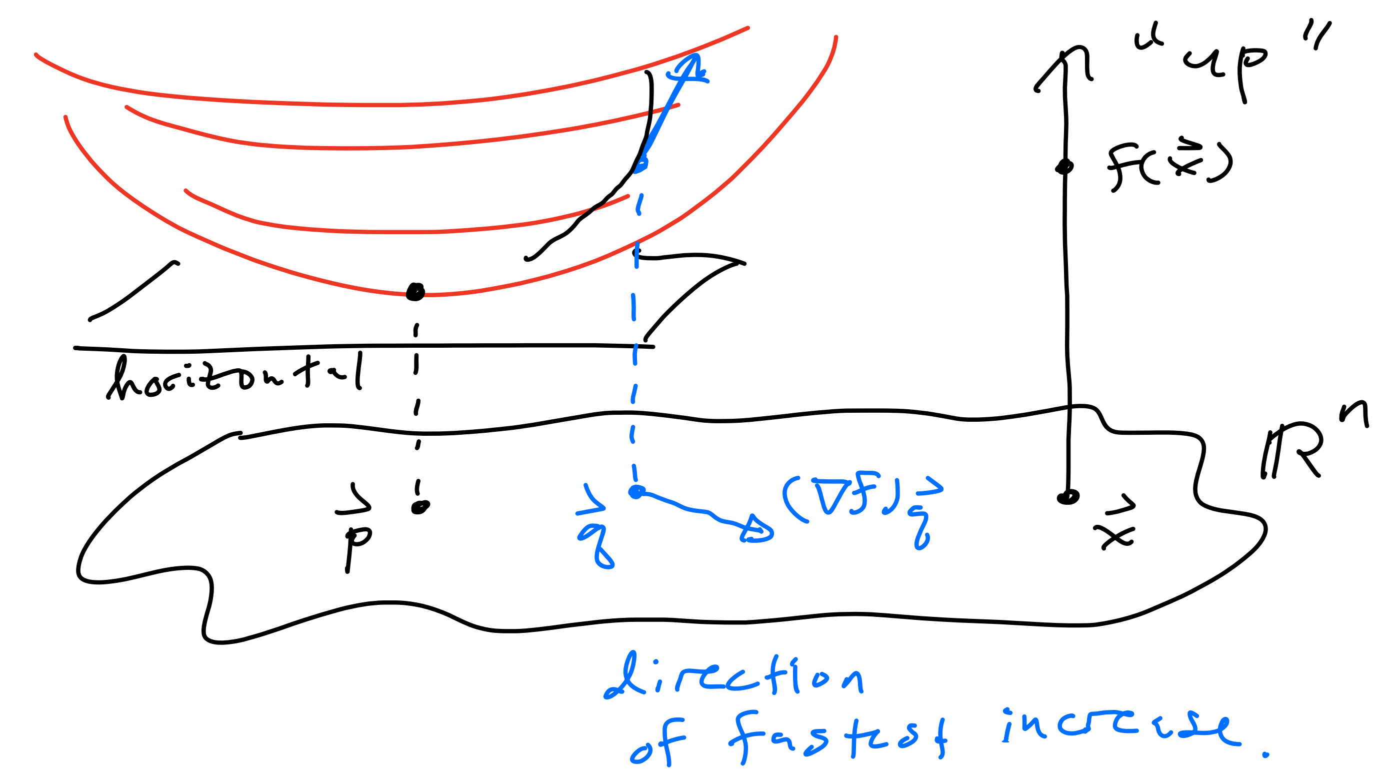
why? $(\nabla f)_{\vec{p}} = (0,0)$

$$f(\vec{p}+\vec{x}) = f(\vec{p}) + 0 + \frac{1}{2}\vec{x}^T (Hf)_{\vec{p}}\, \vec{x}$$

$$= \frac{211}{44} + 0 + \boxed{\frac{1}{2}(x \;\; y)\begin{pmatrix} 6 & 2 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}.$$

near a critical point, $f$ behaves like a quadratic form.

Picture:



Near critical point $\vec{p}$:

$$f(\vec{p}+\vec{x}) \approx f(\vec{p}) + \frac{1}{2}\vec{x}^T (Hf)_{\vec{p}}\, \vec{x}.$$

KEY: $f$ has a local min

$\Longleftrightarrow$ $(Hf)_{\vec{p}}$ is pos-def.

e.g.

$$f(\vec{p}+\vec{x}) = \frac{211}{44} + \frac{1}{2}(x\ \ y)\begin{pmatrix} 6 & 2 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$B = \frac{1}{2}\begin{pmatrix} 6 & 2 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$$

Is this pos. def?

$B = A^T A$ for $A$ with independent columns?

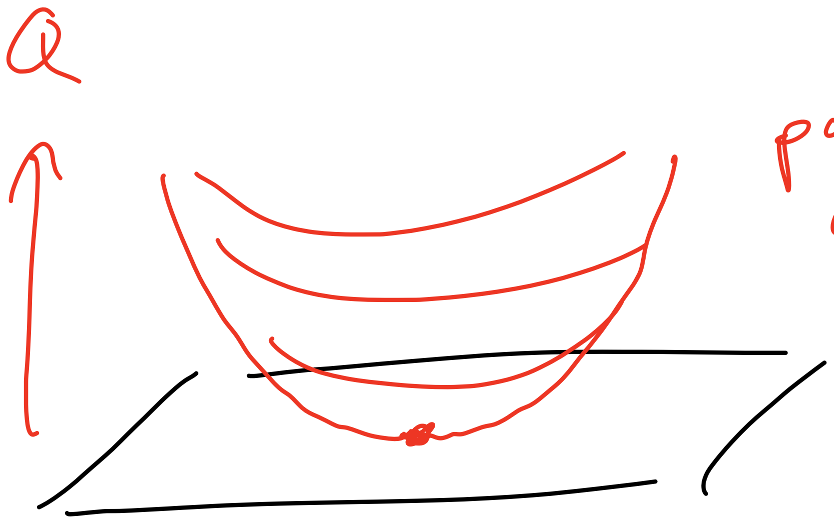
e.g. $2\times 2$ matrices

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is neg. def.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is pos def.

$$Q(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= x^2 + y^2.$$



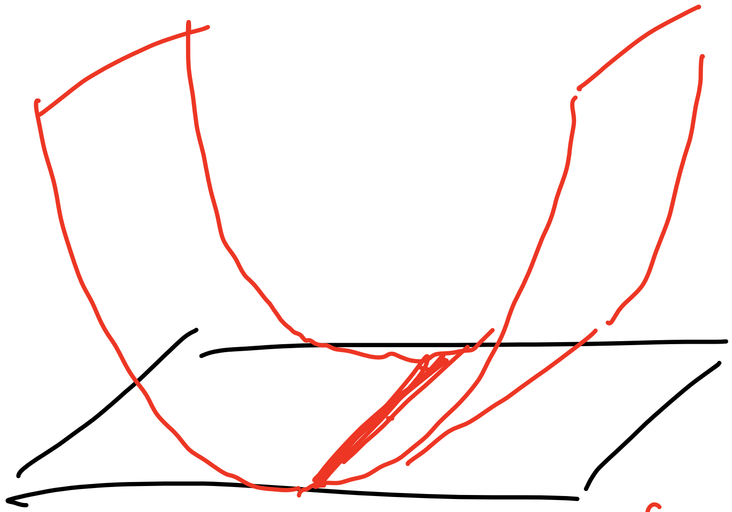

paraboloid.

$Q = x^2 + y^2$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is pos. semi-def.

$$Q(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= x^2$$



whole line of minima.

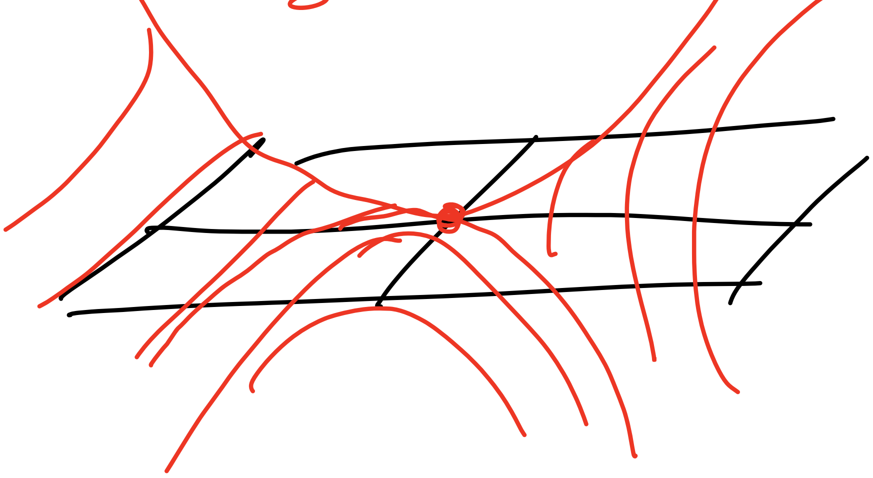
Minimum not unique.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Q(x,y) = (x \ y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= x^2 - y^2.$$

saddle.



Both pos. & neg. values near $(0,0)$

called "indefinite".

$\begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$ is pos. def. because all of its eigenvalues are positive.

What about the higher terms?

$f =$ constant + lin form

+ quad form

+ ? "tensors"

Given vector space $V/\mathbb{R}$,

multilinear k-form
(also called k-tensor) is func

$$\varphi : V^k \to \mathbb{R}$$

k-vectors $\longrightarrow$ scalar
which is linear in each input.

$$\varphi\left(\sum a_i \vec{u}_i, \vec{v}_2, \dots, \vec{v}_n\right)$$

$$= \sum a_i \, \varphi(\vec{u}_i, \vec{v}_2, \dots, \vec{v}_n).$$

Let $\mathcal{T}^k(V)$ be the space of k-tensors. Have seen

$\mathcal{T}^1(V) \longleftrightarrow$ row vectors

$\mathcal{T}^2(V) \longleftrightarrow$ square matrices

$\mathcal{T}^3(V) \longleftrightarrow$ cubes of numbers

$\vdots$ hypercubes of numbers.

$\dim \mathcal{T}^1(\mathbb{R}^n) = \dim$ row vecs $= n.$

$\dim \mathcal{T}^2(\mathbb{R}^n) = \dim$ $n \times n$ matrices

$= n^2.$

$\dim \mathcal{T}^k(\mathbb{R}^n) = n^k.$

Next time I'll give you a standard basis for the vector space of $k$-forms.