New Topic:

Multilinear Algebra & Determinants.

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). Linear function

$$\varphi : V \to \mathbb{R}$$

is called a <u>linear form</u> (sometimes a <u>linear functional</u>).

Linear forms $\varphi : \mathbb{R}^n \to \mathbb{R}$:

Let $b_i = \varphi(\vec{e}_i)$, so

$$\varphi(\vec{x}) = \varphi(x_1 \vec{e}_1 + \cdots + x_n \vec{e}_n)$$

$$= x_1 \varphi(\vec{e}_1) + \cdots + x_n \varphi(\vec{e}_n)$$

$$= b_1 x_1 + \cdots + b_n x_n$$

$$= \vec{b}^T \vec{x}$$

Any linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ has form $\varphi_{\vec{b}}(\vec{x}) = \vec{b}^T \vec{x}$ for some unique vector $\vec{b}^T$. Bijection

linear forms $\longleftrightarrow$ vectors.

Notation: $(\mathbb{R}^n)^\vee$ space of linear functionals.

Recap:

Linear form on $\mathbb{R}^n$ = linear function $\varphi: \mathbb{R}^n \to \mathbb{R}$.

Let $(\mathbb{R}^n)^\vee$ = the set of lin. forms.

Then get bijection:

$$(\mathbb{R}^n)^\vee \longleftrightarrow \mathbb{R}^n$$

$$\varphi_{\vec{b}}(\vec{x}) = \vec{b}^T\vec{x} \qquad \vec{b}$$

$\vec{b}^T$ co-vector $\langle \vec{b} |$

$\vec{b}$ vector $| \vec{b} \rangle$

Abstract Version:

V be inner product space over $\mathbb{R}$ (or $\mathbb{C}$). Let $\varphi : V \to \mathbb{R}$ be linear form. Sometimes

$$\varphi(\vec{v}) = \langle \vec{u}, \vec{v} \rangle$$

for some fixed $\vec{u} \in V$, but not always.

Theorem (Riesz Representation):

φ has form $\varphi_{\vec{u}}(\vec{v}) = \langle \vec{u}, \vec{v} \rangle$ $\iff$ φ is continuous with respect to norm $\sqrt{\langle -, - \rangle}$.

[For Hilbert space, e.g., L^2]

A bilinear form is a function

$$\varphi : V \times V \to \mathbb{R} \text{ (or } \mathbb{C}\text{)}$$

satisfying over $\mathbb{C}$ replace with a_i^*

- $\varphi\left(\sum a_i \vec{u}_i, \vec{v}\right) = \sum a_i \varphi(\vec{u}_i, \vec{v})$
- $\varphi\left(\vec{u}, \sum a_i \vec{v}_i\right) = \sum a_i \varphi(\vec{u}, \vec{v}_i)$.

[over $\mathbb{C}$: "sesquilinear"]

Take $V = \mathbb{R}^n$.

Define $b_{ij} = \varphi(\vec{e}_i, \vec{e}_j)$

Then for any $\vec{x} = x_1 \vec{e}_1 + \cdots + x_n \vec{e}_n$

$\vec{y} = y_1 \vec{e}_1 + \cdots + y_n \vec{e}_n$

we get

$$\varphi(\vec{x}, \vec{y}) = \sum b_{ij} x_i y_j$$

$$= \vec{x}^T B \vec{y} =: \varphi_B(\vec{x}, \vec{y})$$

the matrix of the bilinear form.

Special Cases:

- Symmetric

$$\varphi_B(\vec{x},\vec{y}) = \varphi_B(\vec{y},\vec{x}) \iff B^T = B.$$

Over $\mathbb{C}$:

$$\varphi_B(\vec{x},\vec{y}) = \varphi_B(\vec{y},\vec{x}) \iff B^* = B$$

- Skew-symmetric

$$\varphi_B(\vec{x},\vec{y}) = -\varphi_B(\vec{y},\vec{x}) \iff B^T = -B$$

$$\varphi_B(\vec{x},\vec{y}) = -\varphi_B(\vec{y},\vec{x}) \iff B^* = -B.$$

- Positive-Semidef:

$B = A^* A$

If $B = A^T A$ then

$$\begin{aligned}\varphi_B(\vec{x},\vec{x}) &= \vec{x}^T B \vec{x} \\ &= \vec{x}^T A^T A \vec{x} \\ &= (A\vec{x})^T (A\vec{x})\end{aligned}$$

$$= \|A\vec{x}\|^2 \geq 0.$$

[And conversely, but that's harder to prove ...]

- Pos. <u>Definite</u>:

IF $B = A^T A$, A independent columns,

$$\varphi_B(\vec{x}, \vec{x}) = 0 \iff \vec{x} = \vec{0}.$$

Proof: Say $\varphi_B(\vec{x}, \vec{x}) = 0$.

$$0 = \varphi_B(\vec{x}, \vec{x})$$

$$\vdots$$

$$= \|A\vec{x}\|^2 \iff A\vec{x} = \vec{0}.$$

IF A has ind cols, then

$$A\vec{x} = \vec{0} \iff \vec{x} = \vec{0}.$$

[$A\vec{x} = \vec{0} \Rightarrow (A^T A)^{-1} A^T A \vec{x} = (A^T A)^{-1} A^T \vec{0}$
$\vec{x} = \vec{0}$.]

Bilinear forms $\longrightarrow$ Quadratic forms

$$\varphi_B(\vec{x},\vec{y}) \longrightarrow Q_B(\vec{x}) := \varphi_B(\vec{x},\vec{x}).$$

Abstract Version: Given linear operator $B: V \to V$, get a bilinear function

$$\varphi_B(\vec{x},\vec{y}) = \langle \vec{x}, B\vec{y} \rangle$$

$$\langle \vec{x} | B | \vec{y} \rangle$$

Most Important:

B is an "energy operator"

$\vec{x}$ is a point in "state space"

$$Q_B(\vec{x}) = \langle \vec{x}, B\vec{x} \rangle$$

$$= \langle \vec{x} | B | \vec{x} \rangle$$

= energy density.

$\int Q_B(\vec{x})$ = total energy.

Actual computations use discrete approximation & matrices.

Polynomial of Degree 2

$$F(x,y) = 3x^2 + 2xy + 3y^2 + x - y + 2$$

constant

linear form

$$= 2 + \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

quadratic form.

easy to make this symmetric

$$3x^2 + 2xy + 3y^2$$

$$= 3x^2 + xy + yx + 3y^2$$

$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

More generally:

$$F(x,y,z) = b + b_1 x + b_2 y + b_3 z$$
$$+ b_{11} x^2 + b_{22} y^2 + b_{33} z^2$$
$$+ b_{12} xy + b_{13} xz + b_{23} yz.$$

$$= b + \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$+ \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} b_{11} & \frac{1}{2} b_{12} & \frac{1}{2} b_{13} \\ \frac{1}{2} b_{12} & b_{22} & \frac{1}{2} b_{23} \\ \frac{1}{2} b_{13} & \frac{1}{2} b_{23} & b_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

+ higher terms
(ternary, quaternary ...)
represented by "tensors"

Next: Taylor Series.

$f: \mathbb{R}^n \to \mathbb{R}$

$$
\begin{aligned}
f(\vec{p} + \vec{x}) = {} & f(\vec{p}) \\
& + (\nabla f)_{\vec{p}} \, \vec{x} \\
& + \frac{1}{2} \vec{x}^T (Hf)_{\vec{p}} \, \vec{x} \\
& + \cdots
\end{aligned}
$$