HW 3 Discussion :

CMR Factorization.

Given $m \times n$ $A$,

choose any $r$ ind cols $\vec{c}_1, \dots, \vec{c}_r$,

$r$ ind rows $\vec{r}_1, \dots, \vec{r}_r$.

$$C = \begin{pmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_r \\ | & & | \end{pmatrix} \quad \& \quad R = \begin{pmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_r^{\,T} \end{pmatrix}.$$

e.g. $A = \begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix}$ rank 2.

$$C = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \& \quad R = \begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \end{pmatrix}$$

Claim: $\exists$ invertible $2 \times 2$ $M$ so

$$A = CMR.$$

<u>Existence</u> is harder than I

thought! Assuming $M$ exists,

it is easy to compute:

$$CMR = A$$

$$\underbrace{C^TC}\, M\, \underbrace{RR^T} = C^TAR^T$$

invertible!

$$M = \underbrace{(C^TC)^{-1}C^TAR^T(RR^T)^{-1}}$$

It is not at all obvious to me that this formula has the desired properties!

e.g. $CMR$

$$= C(C^TC)^{-1}C^TAR^T(RR^T)^{-1}R$$

$$= A\ ???$$

In the notes I give an algorithmic

proof that $M$ exists. (Don't read it!)

Examples:

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \quad \text{rank } 1.$$

$$C = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad R = \begin{pmatrix} 2 & 4 \end{pmatrix}$$

$$C^T C = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 4 + 9 = 13$$

$$R R^T = \begin{pmatrix} 2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 4 + 16 = 20.$$

$$M = (C^T C)^{-1} C^T A R^T (R R^T)^{-1}$$

$$= \frac{1}{13} \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} \cancel{2} & \cancel{4} \\ \cancel{3} & \cancel{6} \end{pmatrix} \begin{pmatrix} \cancel{2} \\ \cancel{4} \end{pmatrix} \frac{1}{20}$$

$$= \frac{1}{260} \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 20 \\ 30 \end{pmatrix}$$

$$= \frac{1}{260} (40 + 90) = \frac{130}{260} = \frac{1}{2}.$$

Conclusion

$$A = CMR$$

$$\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 2 & 4 \end{pmatrix}$$

Every rank 1 matrix $A$ factors as $A = \text{col} \cdot \text{row}$

Apply formula

Next:

$$\underbrace{\begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix}}_{\text{rank } 2} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}}_{\text{rank } 2} \begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \end{pmatrix}$$

BIG SURPRISE:

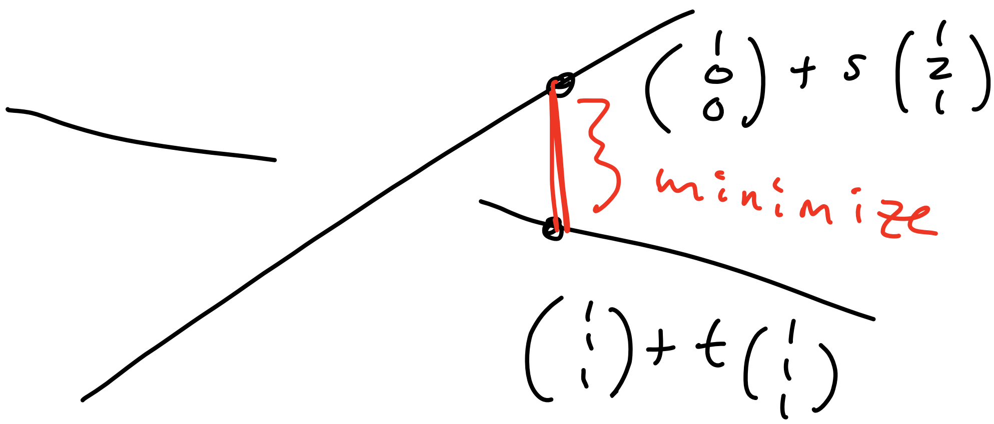
$M^{-1}$ = intersection of columns of $C$ & rows of $R$.

$$= \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$$

e.g.

$$\begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 1 & 6 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix}.$$

Distance Between Skew Lines:




Optimistic: NO THINKING.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad A\vec{x} = \vec{b}$$

NO SOLUTION!

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$A^TA\vec{x} = A^T\vec{b}$$

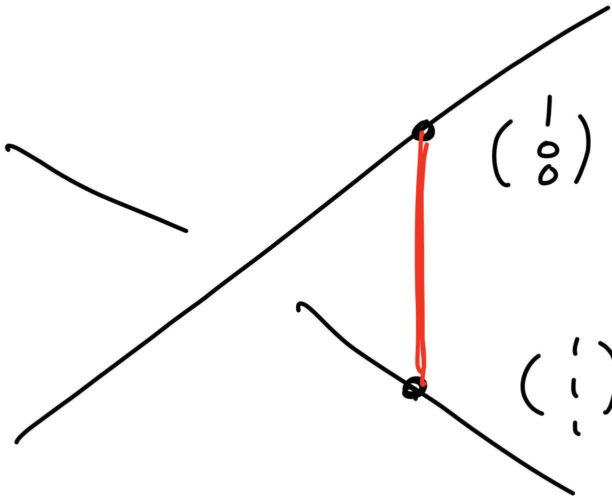
$$\begin{pmatrix} 6 & -4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -4 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$= \frac{1}{18-16} \begin{pmatrix} 3 & 4 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$



$$\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}3/2\\1\\1/2\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}\frac{1}{2}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} + 0\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

Projection Matrices:

Given $\underset{m\times n}{A}$ with ind columns, so $(A^TA)^{-1}$ exists, consider

$$P = \underset{m\times n}{A}\underset{n\times n}{(A^TA)^{-1}}\underset{n\times m}{A^T} \quad m\times m.$$

$$P^2 = A\cancel{(A^TA)^{-1}(A^TA)}(A^TA)^{-1}A^T$$

$$= A(A^TA)^{-1}A^T = P \quad \checkmark$$

"idempotent"

$$P^T = \left[A(A^TA)^{-1}A^T\right]^T$$

$$= A^{TT}\left[(A^TA)^{-1}\right]^T A^T$$

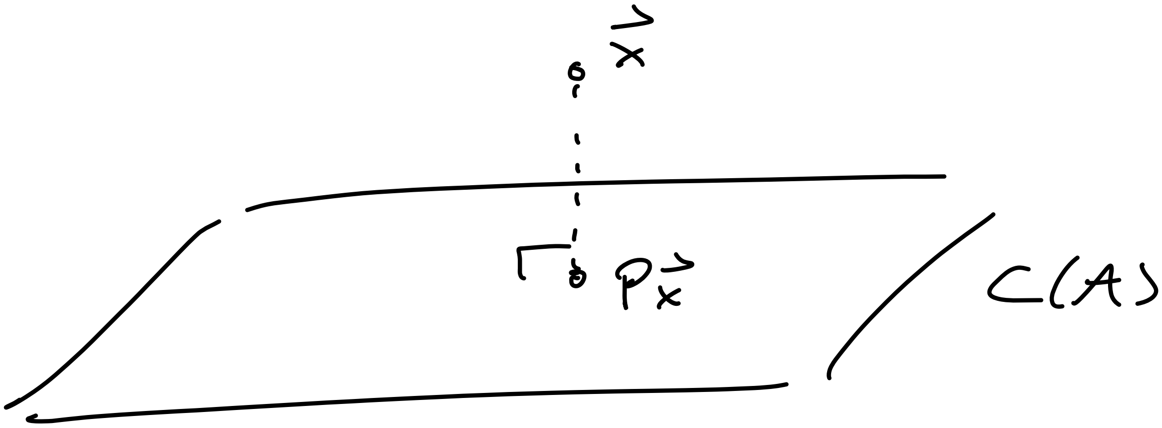
$$= A\left[(A^TA)^T\right]^{-1} A^T$$

$$= A(A^TA)^{-1}A^T = P. \checkmark$$

Hence this $P$ is a projection matrix. Furthermore, one can show that ANY $P$ satisfying

$$P^2 = P \ \& \ P^T = P$$

has the form $P = A(A^TA)^{-1}A^T$ for some matrix $A$. This $P$ is the proj onto $C(A)$.




- $(P\vec{x} - \vec{x}) \perp C(A)$

i.e. $A^T(P\vec{x} - \vec{x}) = \vec{0}$

- $P\vec{x} = A(\text{something}) = A\hat{x}$

Hence
$$A^T(P\vec{x} - \vec{x}) = \vec{0}$$
$$A^T(A\hat{x} - \vec{x}) = \vec{0}$$
$$A^TA\hat{x} - A^T\vec{x} = \vec{0}$$
$$A^TA\hat{x} = A^T\vec{x}$$
$$\hat{x} = (A^TA)^{-1}A^T\vec{x}$$
$$A\hat{x} = A(A^TA)^{-1}A^T\vec{x}$$
$$P\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

e.g. Project onto plane $x - 2y + z = 0$.

TWO WAYS:

- Trick: Project onto $\perp$ line $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$$P = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Bigg[ \underbrace{\begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}_{6} \Bigg]^{-1} \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

Project onto plane:

$$Q = I - P.$$

$$= \frac{1}{6} \begin{pmatrix} 6 & & \\ & 6 & \\ & & 6 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$

- Long way: Pick a basis for the plane

$$A = \begin{pmatrix} | & | \\ | & | \end{pmatrix}$$

two basis vectors for $x - 2y + z = 0$.