HW 3 Help Session, but I won't write down the solutions!

Matrices A^TA & AA^T :

KEY: $N(A^TA) = N(A)$.

Remark: IF E^{-1} exists then already know

$$R(EA) = R(A).$$

$$N(EA)^{\perp} = N(A)^{\perp}$$

$$N(EA) = N(A)$$

But A^T need not be invertible!

need not be square!

$\boxed{\text{TRICK}}$: $\vec{x}^T A^T A \vec{x}$

$$= (A\vec{x})^T (A\vec{x})$$

$$= (A\vec{x}) \cdot (A\vec{x}) = \|A\vec{x}\|^2 .$$

Follows that

$$\dim N(A^T A) = \dim N(A)$$

$$\underset{?}{\rightsquigarrow} \quad \operatorname{rank}(A^T A) = \operatorname{rank}(A)$$

Then know

- $A^T A$ square
- the rank of $A^T A$

$\rightsquigarrow$ does $(A^T A)^{-1}$ exist ?

To study AA^T, instead of doing it from scratch, just replace A by A^T.

CMR Factorization:

Let A be $m \times n$, rank r.

$r = \dim C(A) = \dim R(A)$.

General Fact: Every spanning set of a vector space contains a basis.

Proof: Let $\vec{v}_1, \dots, \vec{v}_k \in V$ be a spanning set for V.

If say $\vec{v}_i = \sum_{j \neq i} c_j \vec{v}_j$ with c_j not all zero, then throw $\vec{v}_i$ away

$$\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \vec{v}_k \in V$$

is still a spanning set. Keep throwing away $\vec{v}$'s until you have an independent spanning set, i.e., a basis. ✓

Consequence: We can find columns $\vec{c}_1, \dots, \vec{c}_r \in \mathbb{R}^m$ & rows $\vec{r}_1, \dots, \vec{r}_r \in \mathbb{R}^n$ so

$\vec{c}_1, \dots, \vec{c}_r$ basis for $C(A)$

$\vec{r}_1, \dots, \vec{r}_r$ basis for $R(A)$

Define
$$C = \begin{pmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_r \\ | & & | \end{pmatrix} \quad m \times r$$

$$R = \begin{pmatrix} - & \vec{r}_1^{\,T} & - \\ & \vdots & \\ - & \vec{r}_r^{\,T} & - \end{pmatrix} \quad r \times n$$

e.g. $A = \begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix} \qquad r = 2$

[Because 3 is a small number the first two cols & rows are bases. For larger matrices, better to compute RREF:

$$\text{RREF}(A) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Definitely the first two cols of A will be a basis for $C(A)$.

Reason: Row ops. preserve col. relations.

$$2(\text{col } 1) + 2(\text{col } 2) = (\text{col } 3).$$

$$2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix}$$

$$2\begin{pmatrix}1\\1\\0\end{pmatrix} + 2\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}8\\6\\2\end{pmatrix} \checkmark$$

A BASIS.

Next: $\underset{m \times r}{C}\ \underset{r \times n}{R} \rightsquigarrow m \times n$

$CR = A$? NO.

$$A = CMR$$

for some (invertible) $r \times r$ M.

$$\begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \end{pmatrix}$$

?

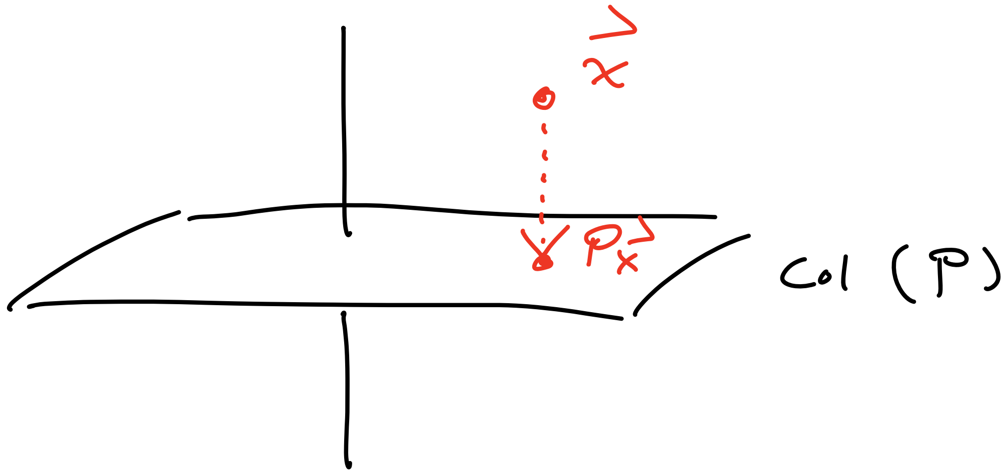
To find M, use the fact that

$(C^T C)^{-1}$ & $(R R^T)^{-1}$ exist,

which follows from Prob 1.

Projection Matrices:

$$\underbrace{P^2 = P}_{\substack{\text{do } P \text{ twice} \\ = \text{ do } P \text{ once}}} \quad \& \quad \underbrace{P^T = P}_{\Rightarrow \text{ square}}$$



$r = \text{rank } P$

P is $n \times n$

Then P is projection onto r-dim subspace in $\mathbb{R}^n$

WHY? Facts:

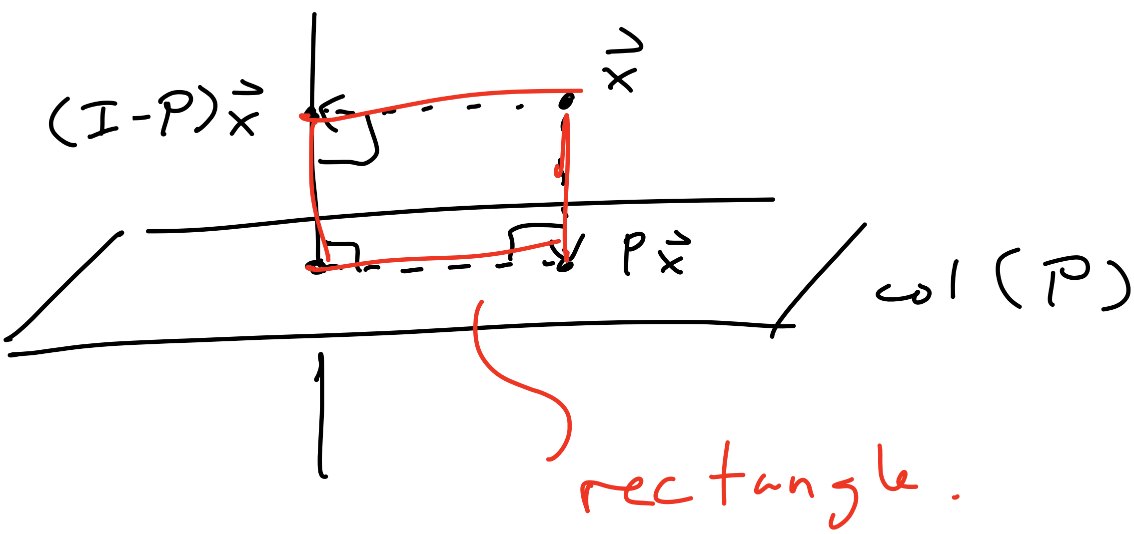
Let $P^2 = P$ & $P^T = P$.

Then $(I-P)^2 = I-P$

$(I-P)^T = I-P$

so $I-P$ also a projection.

Projections come in Pairs!



Indeed:

$$P\vec{x} + (I-P)\vec{x}$$

$$P\vec{x} + \vec{x} - P\vec{x} = \vec{x} \quad ✓$$

Also $(P\vec{x})^T((I-P)\vec{x}) = 0$ ✓

[Behind the scenes:

$$C(P)^\perp = C(I-P). \quad]$$

In general:

$P^2 = P$ P $n \times n$

$P^T = P$

rank $P = r$

see below $\exists$ $n \times r$ A with ind columns

such that $P = A(A^TA)^{-1}A^T$.

Don't prove this.

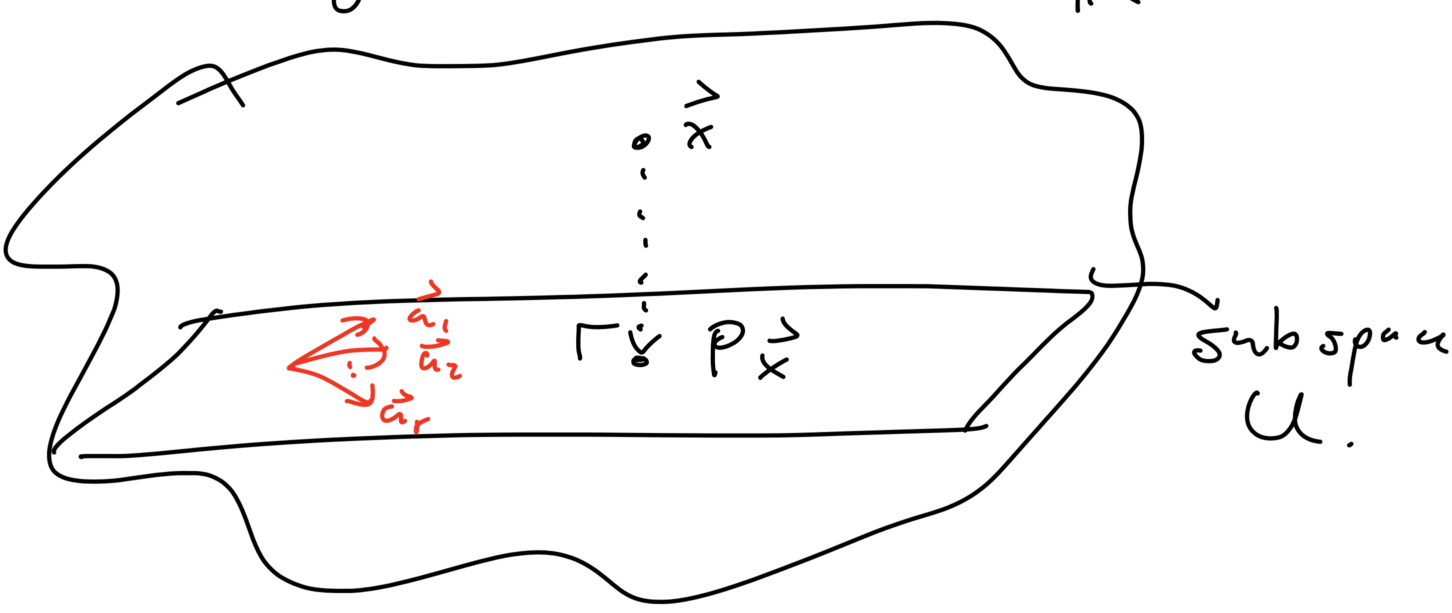
Just check any matrix

$$P = A(A^TA)^{-1}A^T$$

satisfies $P^2 = P$, $P^T = P$.

Moral: Working with projections is mostly algebraic.

Least squares



Let P be projection onto subspace $U \subseteq \mathbb{R}^n$. Say $\dim U = r$.

Let $\mathcal{U} = C(A)$

A $n \times r$ with ind. cols giving a basis of $\mathcal{U}$.

$$A = \underbrace{\begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_r \\ | & & | \end{pmatrix}}_{r} \Big\} n$$

Geometry:

$$P\vec{x} - \vec{x} \perp \text{ all } \vec{a}_i$$

Algebra:

Think $\begin{cases} \circ \quad A^T(P\vec{x} - \vec{x}) = \vec{0} \\ \\ \circ \quad P\vec{x} = A\hat{x} \end{cases}$

for some $\hat{x}$ <u>because</u>

$P\vec{x} \in C(A)$

$\hat{x} \neq \vec{x}$

Put it together:

I don't care what it is!

Don't Think

$$A^T(P\vec{x} - \vec{x}) = \vec{0}$$

$$A^T(A\hat{x} - \vec{x}) = \vec{0}$$

$$A^TA\hat{x} - A^T\vec{x} = \vec{0}$$

$$A^TA\hat{x} = A^T\vec{x}$$

Prob 1

$$\hat{x} = (A^TA)^{-1}A^T\vec{x}$$

$$A\hat{x} = A(A^TA)^{-1}A^T\vec{x}$$

$$P\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

for all $\vec{x}$

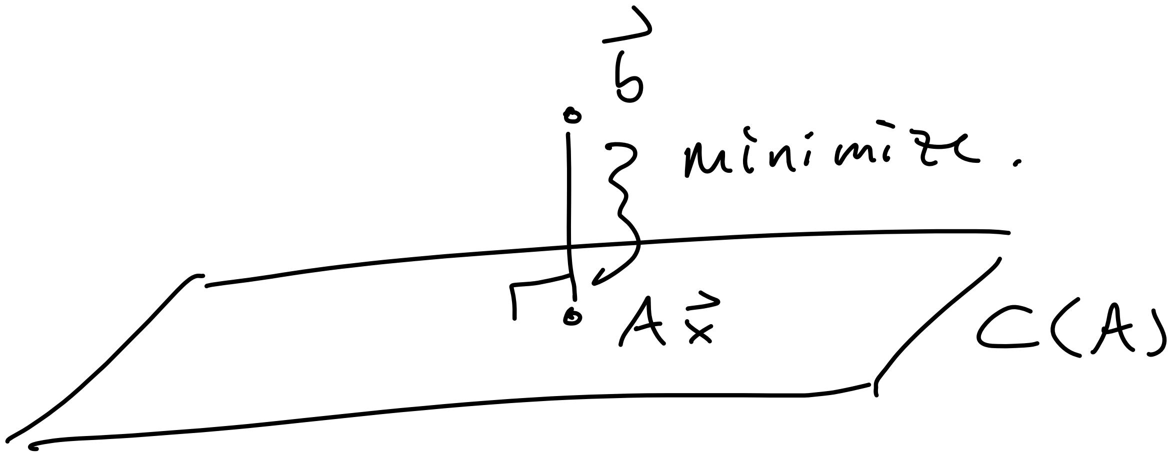
$$\rightsquigarrow \quad P = A(A^TA)^{-1}A^T \quad \checkmark$$

Least squares:

$$A\vec{x} = \vec{b}$$

$\vec{b} \notin C(A)$ so <u>no</u> solution.

Minimize $\|A\vec{x}-\vec{b}\|^2$.



Solution: Let $A\vec{x}$ be <u>projection</u> of $\vec{b}$ onto $C(A)$

$$A\vec{x} = P\vec{b}$$

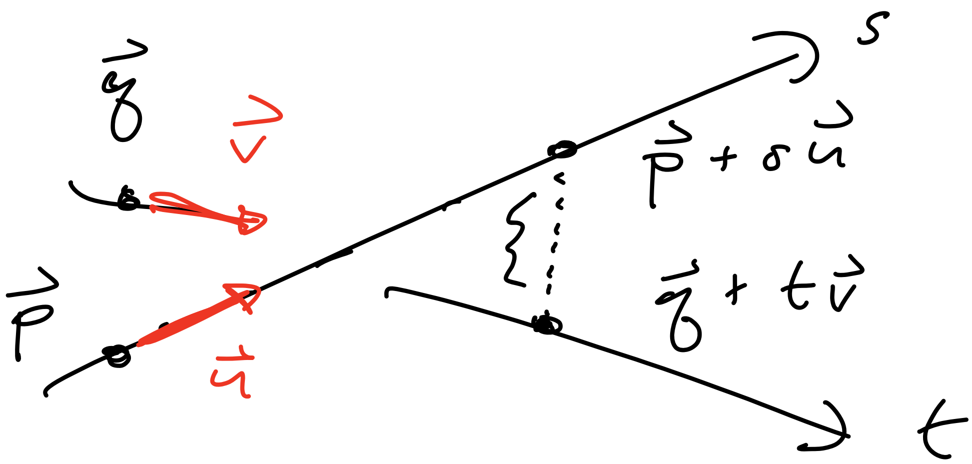
$$A\vec{x} = A(A^TA)^{-1}A^T\vec{b}$$

A not invertible, but still have

$$\vec{x} = (A^TA)^{-1}A^T\vec{b}.$$

least squares solution of $A\vec{x}=\vec{b}$

Distance Between Skew Lines:



Minimize $\|(\vec{p}+s\vec{u})-(\vec{q}+t\vec{v})\|^2$.

Two ways:

1. Calculus
2. Linear Algebra

Idea: Be Optimistic!

$$\vec{p}+s\vec{u}=\vec{q}+t\vec{v}$$
$$s\vec{u}-t\vec{v}=\vec{q}-\vec{p}$$

$$\begin{pmatrix} | & | \\ \vec{u} & -\vec{v} \\ | & | \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = \vec{q} - \vec{p}$$

$$A\begin{pmatrix} s \\ t \end{pmatrix}$$

$$\begin{pmatrix} s \\ t \end{pmatrix} = (A^T A)^{-1} A^T (\vec{q} - \vec{p})$$

See:

$$A^T = \begin{pmatrix} \vec{u}^T \\ -\vec{v}^T \end{pmatrix}$$

$$A^T A = \begin{pmatrix} \vec{u}^T \\ -\vec{v}^T \end{pmatrix}\begin{pmatrix} \vec{u} & -\vec{v} \end{pmatrix}$$

$$= \begin{pmatrix} \|\vec{u}\|^2 & -\vec{u}\cdot\vec{v} \\ -\vec{u}\cdot\vec{v} & \|\vec{v}\|^2 \end{pmatrix}$$

$$\det(A^T A) = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\cdot\vec{v})^2$$

$$= \|\vec{u}\|^2\|\vec{v}\| - (\|\vec{u}\|\|\vec{v}\|\cos\theta)^2$$

$$= \|\vec{u}\|^2\|\vec{v}\|^2(1-\cos^2\theta).$$

$$= \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2\theta$$

From this we can work out a general algebraic formula for s & t. Maybe I'll do this in the typed solutions.