Four Fundamental Subspaces:

Given $m \times n$ matrix A, get 4 subspaces:

$$R(A) \subseteq \mathbb{R}^n$$

$$N(A) \subseteq \mathbb{R}^n \qquad A \vec{x} = \vec{0}$$

$$C(A) \subseteq \mathbb{R}^m \qquad \text{m×n} \quad \text{n×1}$$

$$N(A^T) \subseteq \mathbb{R}^m \qquad \vec{x}^T A = \vec{0}^T$$

Fundamental Theorems:

- Rank-Nullity Theorem

$$\dim R(A) + \dim N(A) = \# \text{cols} = n$$

<u>Proof</u>: $N(A) = R(A)^\perp$

$A\vec{x} = \vec{0} \iff \vec{x} \perp$ every row

$$\begin{pmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_m^T \end{pmatrix} \vec{x} = \begin{pmatrix} \vec{a}_1^T \vec{x} \\ \vdots \\ \vec{a}_m^T \vec{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

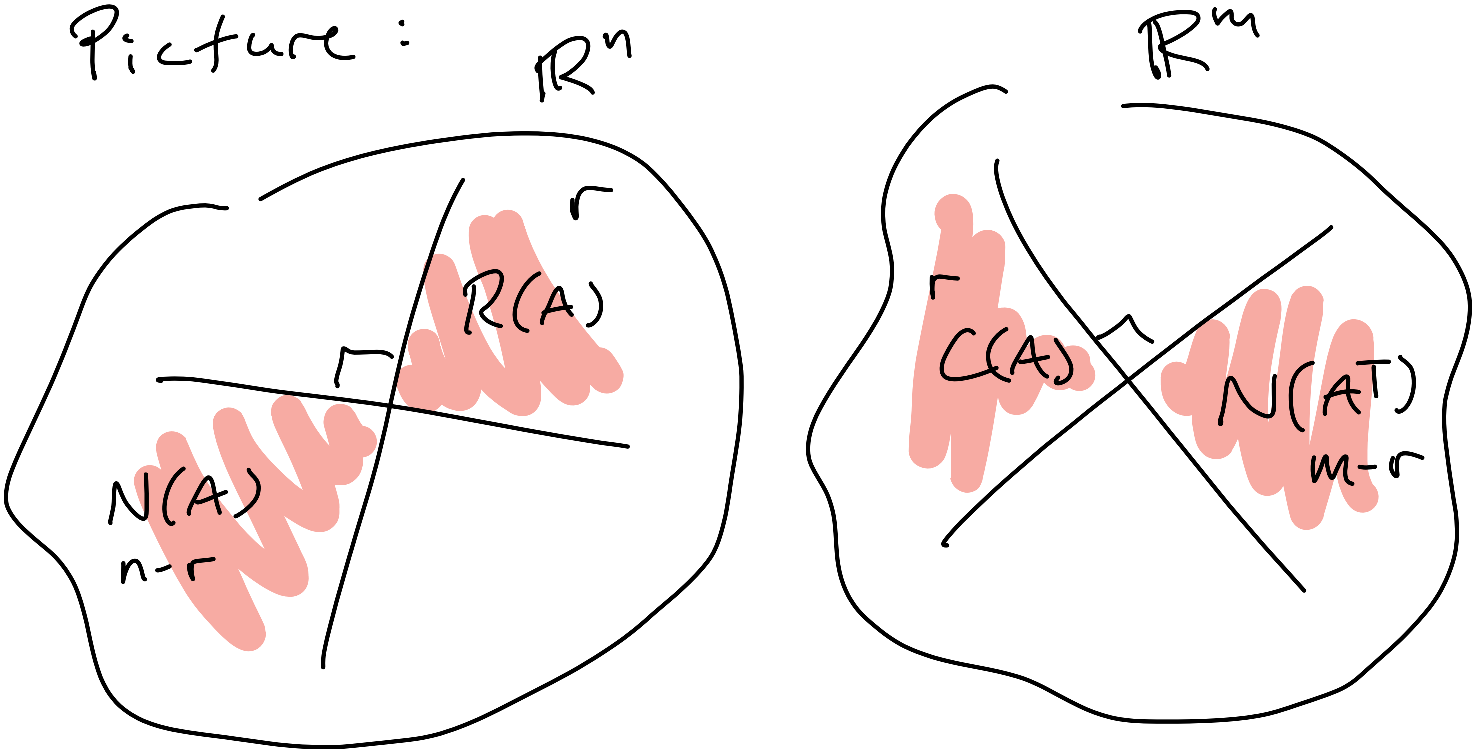
- Similar: $N(A^T) = C(A)^\perp$

- $\dim R(A) = \dim C(A) = r$

- $\dim N(A) = n - r$

  $\dim N(A^T) = m - r.$

Picture:



EXAMPLE:

Consider $A = \begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix}$.

Compute the 4 subspaces.

$$N(A) = \{ \vec{x} : A\vec{x} = \vec{0} \}$$

$$\begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Row reduction:

$$\begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 8 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 3 & 8 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{RREF } \checkmark$$

Matrices:

$$\overbrace{\underbrace{\begin{pmatrix} 1 & -3 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ & +1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{pmatrix}}_{L}}^{E} A$$

upper triang. diagonal lower triang.

$$E = \begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$$EA = \begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

To solve $A\vec{x} = \vec{0}$.

$$EA\vec{x} = E\vec{0}$$

$$R\vec{x} = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x + 2z \\ y + 2z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2z \\ -2z \\ z \end{pmatrix} = \underbrace{z\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}}_{N(A) \text{ is a line.}}$$

$$R(A) = N(A)^{\perp}$$

$= \text{plane} \perp \text{ to line } \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$

$=$ plane with eqn

$$-2x - 2y + z = 0$$

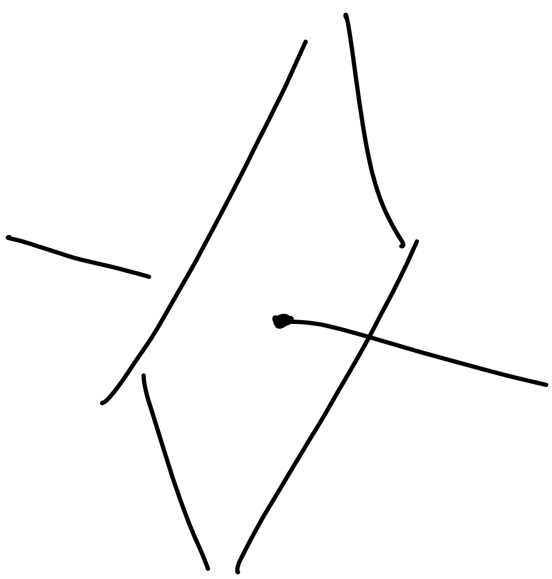
$$2x + 2y - z = 0$$

In fact the rows of R are a basis for $R(A)$ because

$$R(EA) = R(A).$$

$$R\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = R\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 6 \\ 0 & 1 & 8 \end{pmatrix}.$$

$$= s\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$



$N(A)$ line $t\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$

$R(A)$ plane $2x + 2y - z = 0$

Next $C(A)$: TWO WAYS.

(1) $R = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

has pivot cols 1 & 2 so $C(A)$ has basis given by its cols 1 & 2.

basis for $C(A)$ = $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

<u>Proof</u>: Row ops do not change col relations: In R we have.

$$2(\text{col } 1) + 2(\text{col } 2) = (\text{col } 3)$$

Also holds in A:

$$2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 2 \end{pmatrix} \quad \checkmark$$

(2) Compute $R(A^T)$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 8 & 6 & 2 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

so $R(A^T)$ has basis $\begin{pmatrix}1\\0\\1\end{pmatrix}$ & $\begin{pmatrix}0\\1\\-1\end{pmatrix}$

SAME:

$$\text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}.$$

$C(A)$ = plane with eqn ?

$N(A^T)$:
$$\begin{pmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}x+z\\y-z\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-z\\z\\z\end{pmatrix} = z\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$N(A^T)$ = line $t\begin{pmatrix}-1\\1\\1\end{pmatrix}$

$C(A)$ = plane $\perp$ to line $t\begin{pmatrix}-1\\1\\1\end{pmatrix}$

has eqn $-x+y+z=0$.

$x-y-z=0$.

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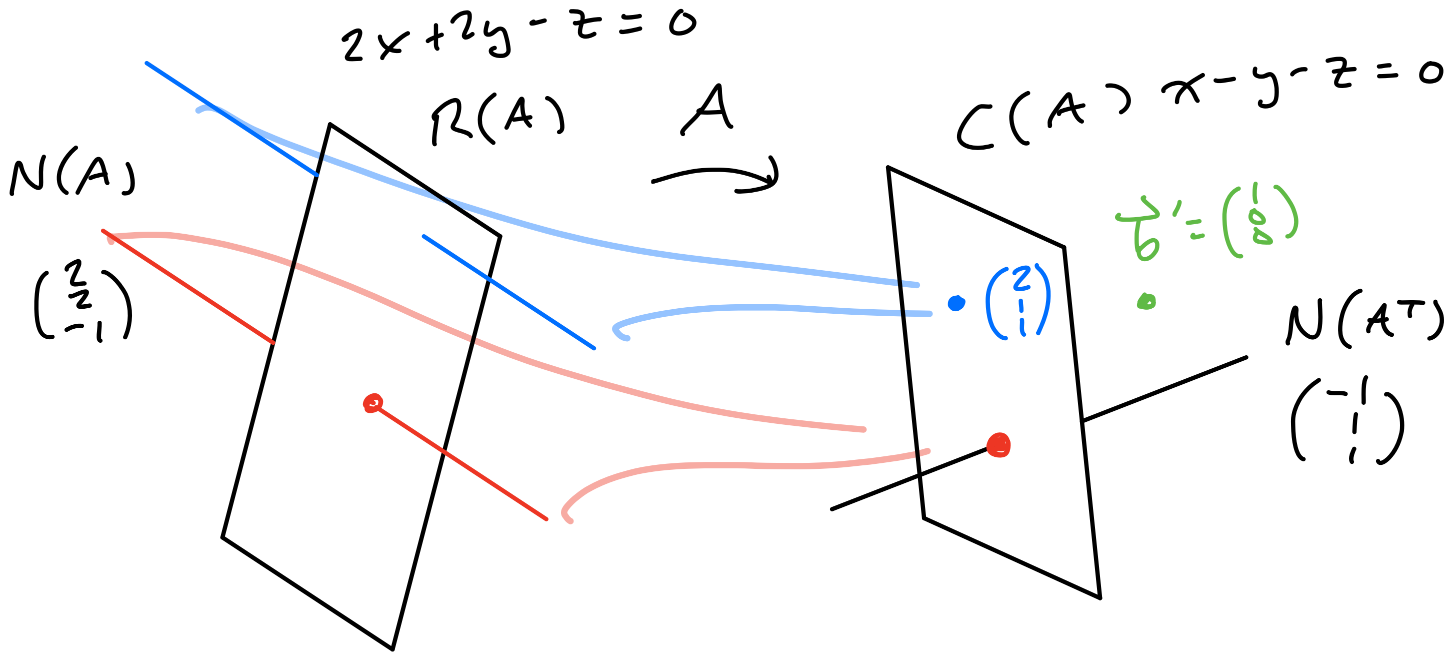

$N(A) \rightsquigarrow$ origin

Note $\begin{pmatrix}2\\1\\1\end{pmatrix}$ is in $C(A)$ since $x-y-z=0$.

So $A\vec{x}=\begin{pmatrix}2\\1\\1\end{pmatrix}$ has a solution!

Solution is a line $\parallel$ to $N(A)$.

$$A\vec{x}=\begin{pmatrix}2\\1\\1\end{pmatrix} \iff \vec{x}=\vec{x}'+t\begin{pmatrix}2\\2\\-1\end{pmatrix}$$

So we just need one particular solution.

$$A\vec{x}=\begin{pmatrix}2\\1\\1\end{pmatrix} \implies EA\vec{x}=E\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$R\vec{x} = \begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{x}'$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1-2z \\ 1-2z \\ 0+z \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

Finally: Take $\vec{b}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin C(A)$.

So $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has NO SOLUTION.

Instead look for $\vec{x}$ so

$$\left\| A\vec{x} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\| \text{ minimized.}$$

Geometrically: Project $\vec{b}'$

onto the col space.