Given square matrix A.

Say λ is an eigenvalue of A

when $\exists\ \vec{x} \neq \vec{0}$ such that

$$\boxed{A\vec{x} = \lambda \vec{x}}$$

$$(\lambda I - A)\vec{x} = \vec{0}.$$

Equivalently:

$$\lambda \text{ e.value} \iff \dim N(\lambda I - A) \geq 1.$$

$$\iff (\lambda I - A)^{-1} \text{ does not exist}$$

$$\iff \underbrace{\det(\lambda I - A) = 0}_{\text{positive statement}}.$$

:)

Characteristic Polynomial:

$$\chi_A(\lambda) = \det(\lambda I - A).$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\chi_A(\lambda) = (\lambda - a)(\lambda - d) - (-b)(-c)$$
$$= \lambda^2 - (a+d)\lambda + (ad-bc).$$

For $n \times n$ matrix:

$\chi_A(\lambda)$ is poly of degree n:

$$\chi_A(\lambda) = \lambda^n - tr(A)\lambda^{n-1} + \cdots + (-1)^n \det(A).$$

where "trace" defined by

$$tr\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = a_{11} + a_{22} + \cdots + a_{nn}.$$

By the Fundamental Theorem of Algebra, $\chi_A(\lambda)$ splits over $\mathbb{C}$.

Meaning: $\exists\, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$

$$\chi_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n),$$

where λ_i might not be distinct!

[In particular, any matrix has at least one eigenvalue in $\mathbb{C}$]

Solving polynomials is a non-linear problem.

If A has n distinct e.values then A has a <u>basis</u> of e.vectors.

<u>Proof</u>: Suppose $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are distinct. Consider e.vectors $\vec{x}_1, \dots, \vec{x}_n$ satisfying

$$A\vec{x}_i = \lambda_i \vec{x}_i \qquad (\vec{x}_i \neq \vec{0}).$$

Claim: $\vec{x}_1, \dots, \vec{x}_n$ are a basis for $\mathbb{C}^n$. Enough to show linear independence. Suppose

$$b_1\vec{x}_1 + \dots + b_n\vec{x}_n = 0. \qquad (1)$$

Goal: $b_1 = b_2 = \cdots = b_n = 0$.

Apply A to ① to get

$$A(b_1\vec{x}_1 + \cdots + b_n\vec{x}_n) = A\vec{0}$$
$$b_1A\vec{x}_1 + \cdots + b_nA\vec{x}_n = \vec{0}$$
$$b_1\lambda_1\vec{x}_1 + \cdots + b_n\lambda_n\vec{x}_n = \vec{0} \qquad ②.$$

Induction on n.

$n=2$: $\quad b_1\vec{x}_1 + b_2\vec{x}_2 = \vec{0} \qquad ①$
$\qquad b_1\lambda_1\vec{x}_1 + b_2\lambda_2\vec{x}_2 = \vec{0} \qquad ②$

② $- \lambda_1$ ①: $\quad 0\vec{x}_1 + b_2\underbrace{(\lambda_2-\lambda_1)}_{\neq 0}\vec{x}_2 = \vec{0}$

Since $\lambda_1 \neq \lambda_2$ & $\vec{x}_2 \neq \vec{0}$

this implies $b_2 = 0$.

Then ①: $\quad b_1\vec{x}_1 + 0\vec{x}_2 = \vec{0}$.

since $\vec{x}_1 \neq \vec{0}$, this implies $b_1 = 0$.

General Case: Fix i.

Consider ② $- \lambda_i$ ①:

$$0\vec{x}_i + \sum_{j\neq i} b_j(\lambda_j - \lambda_i)\vec{x}_j = \vec{0}$$

Induction: $b_j = 0$ for $j \neq i$.

Then (1): $b_i \vec{x}_i + \sum_{j \neq i} 0 x_j = \vec{0}$

$\Longrightarrow b_i = 0$. ✓

If A has a basis of e.vectors say A is "diagonalizable".

Let $X = (\vec{x}_1 | \vec{x}_2 | \cdots | \vec{x}_n)$ be matrix of e.vectors.

$$A\vec{x}_i = \lambda_i \vec{x}_i$$

$$\Longrightarrow (A\vec{x}_1 | \cdots | A\vec{x}_n) = (\lambda_1 \vec{x}_1 | \cdots | \lambda_n \vec{x}_n)$$

$$AX = X\Lambda$$

$$A(\vec{x}_1 | \cdots | \vec{x}_n) = (\vec{x}_1 | \cdots | \vec{x}_n)\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Further, since cols of X are independent, X^{-1} exists:

$$\boxed{A = X \underline{\Lambda} X^{-1}}$$

$$X^{-1} A X = \underline{\Lambda}$$

Why is this good?

Diagonal matrices are <u>easy</u> to work with.

$$S = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix} \quad \& \quad T = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$$

Then $$S+T = \begin{pmatrix} s_1+t_1 & & \\ & \ddots & \\ & & s_n+t_n \end{pmatrix}$$

$$ST = \begin{pmatrix} s_1 t_1 & & \\ & \ddots & \\ & & s_n t_n \end{pmatrix}$$

"componentwise multiplication"

Infinitely simpler than matrix mult!

In particular

$$S^k = \begin{pmatrix} s_1^k & & \\ & \ddots & \\ & & s_n^k \end{pmatrix}.$$

$$A = X \Lambda X^{-1}$$

$$A^2 = X \Lambda \cancel{X^{-1} X} \Lambda X^{-1}$$

$$= X \Lambda^2 X^{-1}$$

$$A^k = X \Lambda^k X^{-1}$$

$$= X \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} X^{-1}$$

Explicit formula for A^k.

But we can do more:

Let $f(x)$ be any polynomial,

$$f(x) = b_0 + b_1 x + \cdots + b_k x^k.$$

Evaluate f at a matrix:

$$f(A) = \underbrace{b_0 I + b_1 A + \cdots + b_k A^k}_{n \times n \text{ matrix}}$$

Important:

$$A = X \Lambda X^{-1}$$

Polynomial $f(x)$

$$f(A) = X f(\Lambda) X^{-1}$$

Proof: holds for addition & mult.

$$A^k = X \Lambda^k X^{-1}$$

$$(A^k + A^l) = X \Lambda^k X^{-1} + X \Lambda^l X^{-1}$$

$$= X(\Lambda^k + \Lambda^l) X^{-1}. \checkmark$$

If A^{-1} exists we can also allow $f(x)$ with negative powers.

$$A^{-3} + A^{-2} = X(\Lambda^{-3} + \Lambda^{-2}) X^{-1}.$$

Example: Suppose A has distinct e.values $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ with $|\lambda_i| < 1$. Then

$$\sum_{k \geq 0} A^k = X \left(\sum_{k \geq 0} \Lambda^k \right) X^{-1}$$

$$= X \begin{pmatrix} \sum \lambda_1^k & & & \\ & \sum \lambda_2^k & & \\ & & \ddots & \\ & & & \sum \lambda_n^k \end{pmatrix} X^{-1}$$

since $|\lambda_i| < 1$.

$$= X \begin{pmatrix} 1/1-\lambda_1 & & \\ & \ddots & \\ & & 1/1-\lambda_n \end{pmatrix} X^{-1}.$$

on the other hand:

$$(I - A)^{-1} = X (I - \Lambda)^{-1} X^{-1}$$

$$= X \begin{pmatrix} \frac{1}{1-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{1-\lambda_n} \end{pmatrix} X^{-1}$$

$$= \sum_{k \geq 0} A^k$$

Another e.g.

$$A = X \Lambda X^{-1}$$

$$f(x) = e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$e^A = X e^{\Lambda} X^{-1}$$

$$= X \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} X^{-1}$$

[Purpose: Linear system of diff. eq. $\vec{x}'(t) = A\vec{x}(t)$.

has general soln. $\vec{x}(t) = \underset{n\times n}{e^{At}}\,\underset{n\times 1}{\vec{x}(0)}$.

Proof: $\dfrac{d}{dt} e^{At} = A e^{At}$.

Example: Diagonalize $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to show that

$$e^{\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

" $e^{it} = \cos t + i \sin t$ "

☺ : A randomly chosen $n\times n$ matrix will be diagonalizable.

Why? Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\chi_A(\lambda) = \lambda^2 - (a+d)\lambda + (ad-bc).$$

Have distinct roots when

$$\Delta = (a+d)^2 - 4(ad-bc) \neq 0.$$

But $\Delta = 0$ defines a 3D shape in 4D space of 2×2 matrices.

$$\underset{\text{dim } 3}{(\Delta = 0)} \subseteq \underset{\text{dim } 4}{\mathbb{C}^{2\times 2}}$$

A randomly chosen 2×2 matrix will have $\Delta \neq 0$ hence 2 distinct e.values, hence a basis of e.vectors.