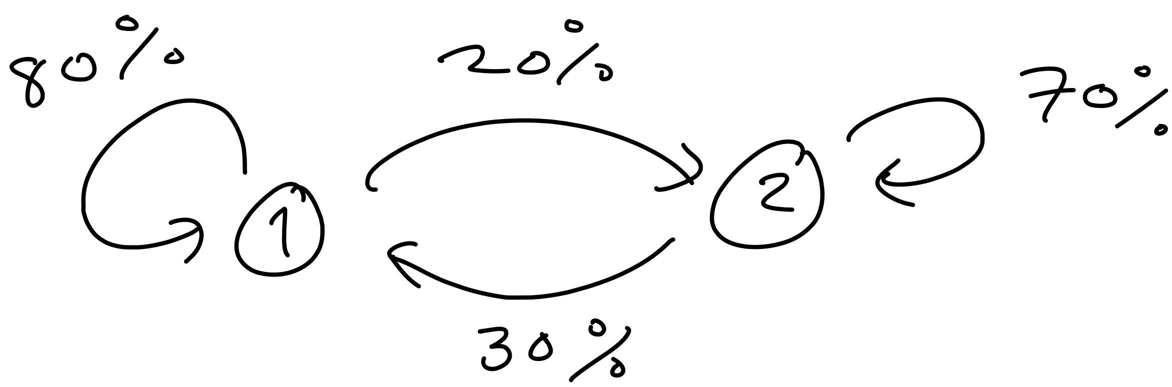
Next Topic: Eigenvalues & Eigenvectors.

Motivating Example:

$$A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$$

(Example of a Markov Chain.)

Say a particle can occupy two possible states, with transition probabilities:



Evolution over time?

Let p_n = probability of being in state 1 at time n.

$q_n = 1 - p_n$
= prob of state 2.

Claim:

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = A \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Proof: Law of Total Probability.

$P(1 \text{ now}) = P(1 \text{ now } \& \ 1 \text{ last time})$
$\quad + P(1 \text{ now } \& \ 2 \text{ last time})$
$= P(1 \text{ last time}) P(1 \text{ now} \mid 1 \text{ last})$
$\quad + P(2 \text{ last}) P(1 \text{ now} \mid 2 \text{ last})$.

$\{ P(S) = P(S \cap T) + P(S \cap T')$
$= P(T) P(S \mid T) + P(T') P(S \mid T'). \}$

So $p_n = p_{n-1}(.8) + q_{n-1}(.3)$

Similarly

$$q_n = p_{n-1}(.2) + q_{n-1}(.7)$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}.$$

Starting distribution $\vec{p}_0 = (p_0, q_0)$.

$$\vec{p}_1 = A\vec{p}_0$$

$$\vec{p}_2 = A\vec{p}_1 = AA\vec{p}_0 = A^2\vec{p}_0$$

$$\vec{p}_3 = \cdots = A^3\vec{p}_0$$

$$\vdots$$

$$\vec{p}_n = A^n\vec{p}_0.$$

Problem: Analyze the powers of A.

$$A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} .7 & .45 \\ .3 & .55 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} .65 & .525 \\ .35 & .475 \end{pmatrix}$$

Pattern?

$$A^{100} \approx \begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix}$$

Looks like $A^n \longrightarrow \begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix}$

$n \longrightarrow \infty$.

How?

Exact formula for A^n?

KEY: Eigenvectors.

Let me just tell you that

$$A\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$A\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}$$

So $A^n\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}$ for any n.

$$A^n\begin{pmatrix}1\\-1\end{pmatrix} = \left(\frac{1}{2}\right)^n\begin{pmatrix}1\\-1\end{pmatrix}$$

Express initial condition $\vec{p}_0$ in terms of eigenvectors.

$$\vec{p}_0 = a\begin{pmatrix}3\\2\end{pmatrix} + b\begin{pmatrix}1\\-1\end{pmatrix}.$$

$$= \begin{pmatrix}3 & 1\\2 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

$$\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}3 & 1\\2 & -1\end{pmatrix}^{-1}\begin{pmatrix}p_0\\q_0\end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix}-1 & -1\\-2 & 3\end{pmatrix}\begin{pmatrix}p_0\\q_0\end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix}-p_0 - q_0\\-2p_0 + 3q_0\end{pmatrix}$$ $p_0 + q_0 = 1.$

$$= \begin{pmatrix} 1/5 \\ p_0 - 3/5 \end{pmatrix}.$$

So $$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \left(p_0 - \frac{3}{5}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now we're done!

$$\vec{p}_n = A^n \vec{p}_0$$
$$= \frac{1}{5} A^n \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \left(p_0 - \frac{3}{5}\right) A^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$= \frac{1}{5}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \left(p_0 - \frac{3}{5}\right)\left(\frac{1}{2}\right)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$p_n = 3/5 + (p_0 - 3/5)/2^n$$
$$q_n = 2/5 - (p_0 - 3/5)/2^n$$

$(1/2)^n \to 0$ as $n \to \infty$.

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} \to \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$$

Independent of p_0 & q_0 !

We can do more :

$$A\left(\begin{array}{c|c} 3 & 1 \\ 2 & -1 \end{array}\right) = \left(A\begin{pmatrix} 3 \\ 2 \end{pmatrix} \middle| A\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$$

$$= \left(\begin{array}{c|c} 3 & 1/2 \\ 2 & -1/2 \end{array}\right)$$

TRICK

$$= \left(\begin{array}{c|c} 3 & 1 \\ 2 & -1 \end{array}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$A S = S \Lambda$$

$$\boxed{A = S \Lambda S^{-1}}$$

eigenvectors

eigenvalues.

We have "diagonalized" A.

$$A^2 = S\Lambda \cancel{S^{-1} S} \Lambda S^{-1}$$

$$= S \Lambda^2 S^{-1}$$

$$\vdots$$

$$A^n = S \Lambda^n S^{-1}$$

$$A^n = \left(\begin{array}{c|c} 3 & 1 \\ 2 & -1 \end{array}\right) \begin{pmatrix} 1 & \\ & 1/2 \end{pmatrix}^n \left(\begin{array}{c|c} 3 & 1 \\ 2 & -1 \end{array}\right)^{-1}$$

$$= \left(\begin{array}{c|c} 3 & 1 \\ 2 & -1 \end{array}\right) \begin{pmatrix} 1^n & \\ & (1/2)^n \end{pmatrix} \left(\begin{array}{c|c} 3 & 1 \\ 2 & -1 \end{array}\right)^{-1}$$

$$= \left(\begin{array}{c|c} 3 & 1 \\ 2 & -1 \end{array}\right) \begin{pmatrix} 1 & \\ & (1/2)^n \end{pmatrix} \left(-\frac{1}{5}\right) \begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix}.$$

$$= \left(\begin{array}{c|c} 3 & 1 \\ 2 & -1 \end{array}\right) \begin{pmatrix} 1 & \\ & (1/2)^n \end{pmatrix} \left(\begin{array}{cc} 1/5 & 1/5 \\ \hline 2/5 & -3/5 \end{array}\right)$$

[TRICK :

$$\begin{pmatrix} | & & | \\ \vec{x}_1 & \cdots & \vec{x}_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} - & \vec{y}_1^{\,T} & - \\ & \vdots & \\ - & \vec{y}_n^{\,T} & - \end{pmatrix}$$

$$= \lambda_1 \vec{x}_1 \vec{y}_1^{\,T} + \cdots + \lambda_n \vec{x}_n \vec{y}_n^{\,T}$$

sum of rank 1 matrices.]

$A^n =$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \end{pmatrix} + \left(\frac{1}{2}\right)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & -\frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{pmatrix} + \left(\frac{1}{2}\right)^n \begin{pmatrix} 2/5 & -3/5 \\ -2/5 & 3/5 \end{pmatrix}.$$

$$\longrightarrow \begin{pmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{pmatrix} \quad \text{as} \quad n \to \infty.$$

Given SQUARE matrix A, say scalar λ is an eigenvalue if there exists a nonzero $\vec{x}$ such that

$$\boxed{A\vec{x} = \lambda\vec{x}}$$

Then we say $\vec{x}$ is a λ-eigenvector.

Do eigenvalues exist?

$$A\vec{x} = \lambda\vec{x}$$
$$\lambda\vec{x} - A\vec{x} = \vec{0}$$
$$\lambda I\vec{x} - A\vec{x} = \vec{0}$$
$$(\lambda I - A)\vec{x} = \vec{0}$$
$$\vec{x} \in \underbrace{N(\lambda I - A)}_{\text{the space of } \lambda\text{-eigenvectors.}}$$

λ is eigenvalue

$\iff \dim N(\lambda I - A) \neq 0.$

$\iff \lambda I - A$ <u>not</u> invertible.

$\iff \det(\lambda I - A) = 0.$

Observe that $\chi_A(\lambda) :=$

$\det(\lambda I - A)$ is a polynomial in λ of degree n.

called "characteristic polynomial of A".

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\chi_A(\lambda) = \det\left(\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix}.$$

$$= (\lambda - a)(\lambda - d) - (-b)(-c)$$

$$= \lambda^2 - (a+d)\lambda + (ad - bc)$$

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

These ~~exist~~ but they might be complex.

e.g. $A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}.$

$$\chi_A(\lambda) = (\lambda - .8)(\lambda - .7) - (-.2)(-.3)$$

$$= \lambda^2 - 1.5\lambda + \underbrace{\cancel{(.8)(.7)}}_{.56} - \underbrace{\cancel{(.2)(.3)}}_{.06}$$

$$= \lambda^2 - 1.5\lambda + 0.5.$$

Eigenvalues:

$$\lambda = \frac{1.5 \pm \sqrt{2.25 - 2}}{2}$$

$$= \frac{1.5 \pm 0.5}{2}$$

$$= 1 \text{ or } 1/2.$$

Once we know the eigenvalues, use row reduction to find bases for eigenspaces.

Finding eigenvalues is a non-linear problem. But there are relatively fast approximation schemes [e.g. QR method]

Some matrices don't have "enough" eigenvectors.

e.g. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\chi_A(\lambda) = (\lambda-1)^2$

only one eigenvalue $\lambda = 1$.

And $N(1I - A)$

$= N\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$

$= N\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

has dim 1.

So this A does not have a basis of e. vectors.

It is not "diagonalizable".