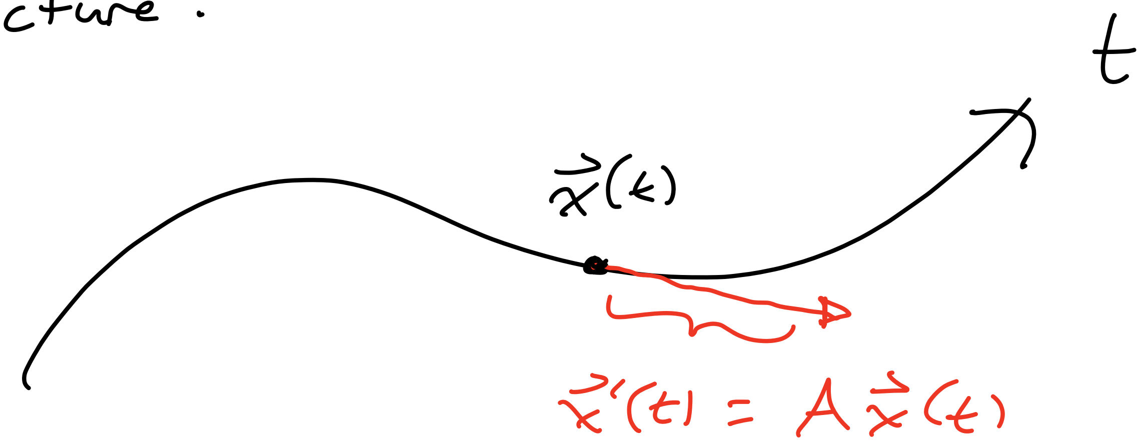
Recall: Given $n \times n$ A consider the dynamical system

$$\vec{x}'(t) = A\vec{x}(t)$$

Picture:



Matrix A is a "vector field".
Solution $\vec{x}(t)$ is a "flow line" of the field.

Solution:

$$\vec{x}(t) = \exp(At)\,\vec{x}(0).$$

<u>Proof</u>: $\dfrac{d}{dt}\left(\exp(At)\,\vec{x}(0)\right)$

$$= A\,\boxed{\exp(At)\,\vec{x}(0)}$$

__Uniqueness__: let $\vec{x}'(t) = A\vec{x}(t)$

be __any__ solution & consider

$$\vec{y}(t) = e^{-At}\,\vec{x}(t).$$

$$\vec{y}'(t) = -A\,e^{-At}\,\vec{x}(t)$$

$$+\, e^{-At}\,\cancel{\vec{x}'(t)}$$

$$A\vec{x}(t)$$

$$= e^{-At}(-A+A)\,\vec{x}(t)$$

$$= \vec{0}.$$

$\Longrightarrow$ $\vec{y}(t)$ is constant.

Put $t=0$, see $\vec{y}(t) = \vec{x}(0)$.

$$\vec{y}(0) = \cancel{e^{-A0}}\,\vec{x}(0) = \vec{x}(0).$$

Hence $\vec{x}(0) = e^{-At}\vec{x}(t)$

$$\Rightarrow \quad \vec{x}(t) = e^{At}\vec{x}(0) \quad \checkmark$$

Thus we need a method to compute matrix exponentials.

Jordan Form:

$$A = X \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix} X^{-1}$$

$$J = \left.\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}\right\} n \times n.$$

$$\exp(Jt) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{n-1}}{(n-1)!}e^{\lambda t} \\ & \ddots & & \vdots \\ & & \ddots & te^{\lambda t} \\ & & & e^{\lambda t} \end{pmatrix}$$

Simplest examples:

Any 2×2 real matrix has form

$$A = X\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}X^{-1}$$

$$\text{or } X\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}X^{-1}$$

complex eigenvalues.

$$\text{or } X\begin{pmatrix} a & -b \\ b & a \end{pmatrix}X^{-1}$$

$$\exp\left(\begin{pmatrix} a & \\ & b \end{pmatrix}t\right) = \begin{pmatrix} e^{a} & \\ & e^{b} \end{pmatrix}$$

$$\exp\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}t\right) = \ ?$$

$$\exp\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}t\right) = \ ?$$

e.g. $\exp\left(\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} t\right)$

$$= \exp\left(\begin{pmatrix} -t & \\ & -t \end{pmatrix} + \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}\right).$$

commute!

$$= \exp\begin{pmatrix} -t & \\ & -t \end{pmatrix} \exp\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}.$$

$$= \begin{pmatrix} e^{-t} & \\ & e^{-t} \end{pmatrix}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

$$= \begin{pmatrix} e^{-t}\cos t & -e^{-t}\sin t \\ e^{-t}\sin t & e^{-t}\cos t \end{pmatrix}.$$

System

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

has solution

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} =$$

$$\begin{pmatrix} e^{-t}\cos t & -e^{-t}\sin t \\ e^{-t}\sin t & e^{-t}\cos t \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

Done.

Amazing Trick:

Use same theorem to solve higher order diff eq.

$$(*) \quad y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots a_{n-1} y'(t) + a_n y(t) = 0.$$

Define a vector $\vec{x}(t) \in \mathbb{R}^n$

$$\vec{x}(t) = (y(t), y'(t), \dots, y^{(n)}(t)).$$

(*) Becomes a linear system:

$$\vec{x}'(t) = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ -a_n & \cdots & \cdots & \cdots & -a_2 & -a_1 \end{pmatrix} \vec{x}(t)$$

The "companion matrix" C

Problem: $\exp(Ct)$ for companion matrices C.

<u>Theorem</u>: Char poly

$$\chi_C(x) = \pm(x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n)$$

$$= \prod_i (x-\lambda_i)^{n_i}$$

tell you the Jordan blocks.

Then Jordan form of C is

$$\begin{pmatrix} J_{n_1}(\lambda_1) & & & \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{n_k}(\lambda_k) \end{pmatrix}$$

Corollary:

Original $\circledast$ has solutions

$$t^j e^{\lambda_i t} \quad (0 \le j \le n_i - 1).$$

Full set of linearly independent solutions.

Markov Chains:

Stochastic matrix M has entries in $[0,1]$ and each col sums to 1.

e.g. $$M = \begin{pmatrix} 1/2 & 1/3 & 0 \\ 1/4 & 0 & 1/2 \\ 1/4 & 2/3 & 1/2 \end{pmatrix}.$$

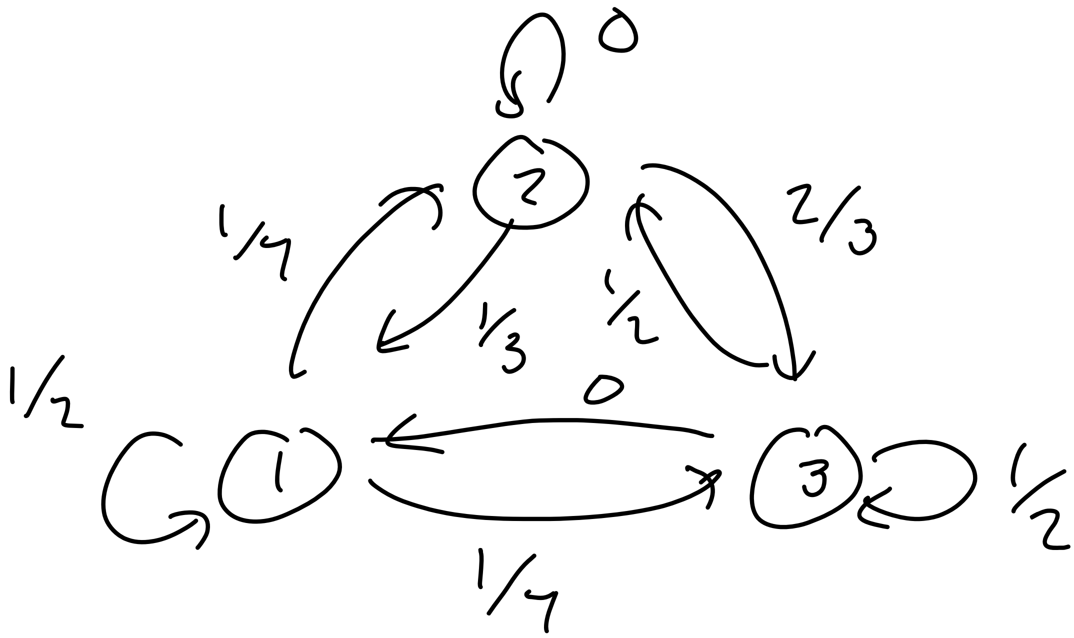
Also called a "Markov chain".

"Perron-Frobenius Theorem".

If some power of M has no zero entries then $\exists$ unique equilibrium distribution $\vec{p}$:

$$M\vec{p} = \vec{p}.$$

Entries of $\vec{p}$ sum to 1.



$\vec{p}$ is the unique equilibrium.