Applications :

Today : Dynamical Systems.

- Discrete :

  Sequence of points

  $\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots$

  Where $\vec{x}_{k+1} = A\vec{x}_k$

  [Example : Markov chain.]

$$\begin{aligned} \vec{x}_n &= A\vec{x}_{n-1} \\ &= AA\vec{x}_{n-2} \\ &\vdots \\ &= A^n\vec{x}_0 \end{aligned}$$

Problem : Analyze powers A^n.

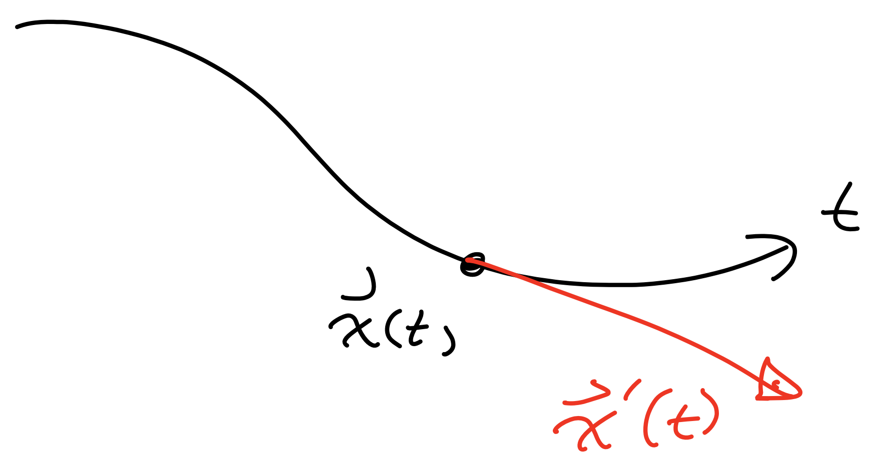
- Continuous Dynamical Systems.

  vector changes with time.

$$\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)).$$

Also think of a parametrized path

$\vec{x} : \mathbb{R} \to \mathbb{R}^n$



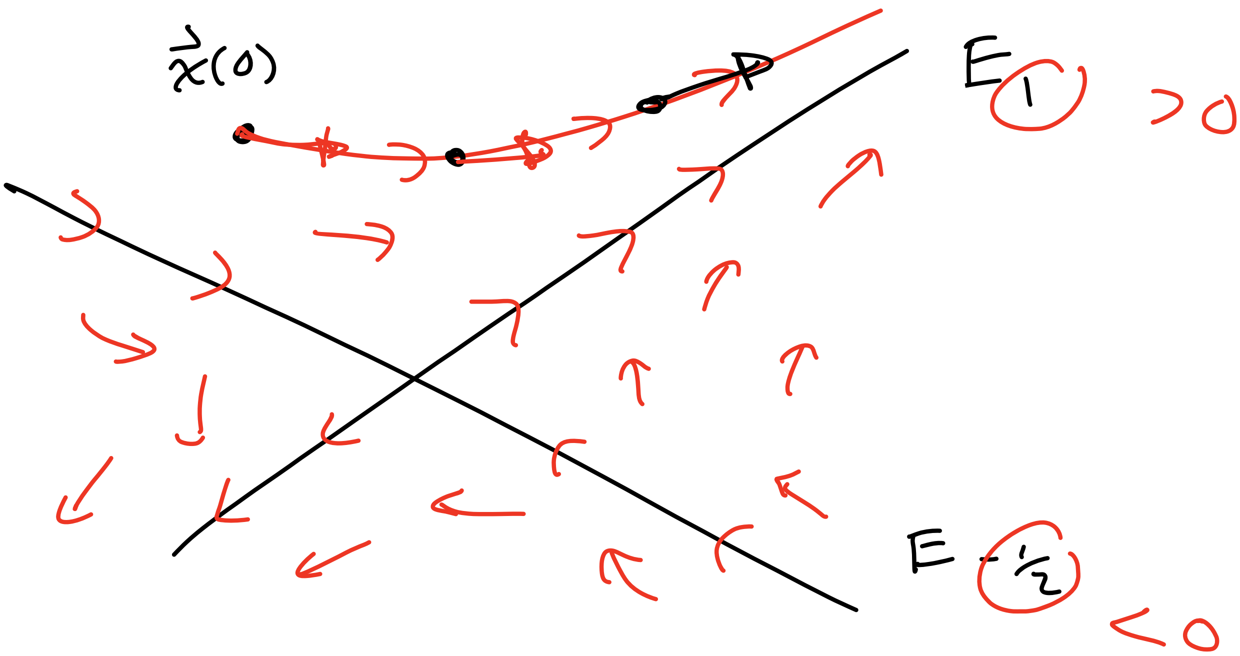
Dynamical system is equation relating velocity & position.

Simplest Example:

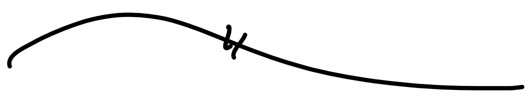
$$\vec{x}'(t) = \underbrace{A\vec{x}(t)}_{\text{think of a vector field.}}$$

At every point $\vec{x}(t)$ there is an arrow $A\vec{x}(t)$. Solution: A path that "flows" along the vector field.

Picture: A is 2×2 with eigenvalues 1 & $-1/2$



Picture depends on nature of eigenvalues



Book Recommendation:

Hirsch & Smale "Intro to Dynamical Systems". Based on linear algebra.

One technicality:

Jordan canonical form
(normal)

3 theorems on non-diag matrices:

- Schur Triangularization

$$A = U T U^*$$

- Singular Value Decomp.

- Jordan form.

For any matrix A, can find

X such that

$$A = X \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix} X^{-1}$$

Jordan block matrix has form

$$J = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

Good: We can still compute

A^n & $\exp(At)$

explicitly using Jordan blocks.

Homework 6: Compute

J^n & $\exp(Jt)$

for Jordan blocks.

Theorem: Dynamical System

$$\vec{x}\,'(t) = A\vec{x}(t)$$

with initial position $\vec{x}(0)$ has

the unique solution

$$\vec{x}(t) = \exp(At) \cdot \vec{x}(0)$$

Proof: Goes back to Euler.

Definition of $\exp(x)$ for scalars

$$\exp(x) = 1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{k!}x^k + \dots$$

Properties:

- $$\frac{d}{dx}\exp(x) = 0 + 1 + \frac{1}{2}2x + \dots + \frac{1}{k!}k\,x^{k-1} + \dots$$

$$\boxed{\frac{k}{k!} = \frac{1}{(k-1)!}} \qquad = \exp(x)$$

Diag'ble matrix

$$A = X \Lambda X^{-1}$$

$$\exp(A) = X \exp(\Lambda) X^{-1}$$

$$\begin{pmatrix} \exp(\lambda_1) & & \\ & \ddots & \\ & & \exp(\lambda_n) \end{pmatrix}$$

- $\exp(x+y) = \exp(x) \cdot \exp(y)$.

Uses binomial theorem.

The same properties for matrices?

$$\exp(A+B) \neq \exp(A) \cdot \exp(B)$$

in general! If $AB = BA$, the proof for scalars shows that

$$\exp(A+B) = \exp(A) \cdot \exp(B).$$

$$\exp(At) = I + tA + \frac{t^2}{2!} A^2 + \cdots + \frac{t^k}{k!} A^k + \cdots$$

$$\underbrace{\frac{d}{dt}\exp(At)}_{\text{componentwise}} = 0 + A + tA^2 + \cdots + \frac{t^{k-1}}{(k-1)!}A^k + \cdots$$

$$= A\left(I + tA + \cdots + \frac{t^{k-1}}{(k-1)!}A^{k-1} + \cdots\right)$$

$$= A \cdot \exp(At).$$

Given $\vec{x}'(t) = A\vec{x}(t)$

with $\vec{x}(0)$. Then claim

$$\vec{x}(t) = \exp(At) \cdot \vec{x}(0)$$

Is the solution.

<u>Proof</u>: $\frac{d}{dt}\exp(At) \cdot \vec{x}(0)$

$$= A\exp(At) \cdot \vec{x}(0)$$

$$= A\left(\exp(At) \cdot \vec{x}(0)\right).$$

At time $t=0$

$$\exp(A\cdot 0)\,\vec{x}(0)$$

$$\exp(0)\cdot\vec{x}(0)$$

$$I\cdot\vec{x}(0) = \vec{x}(0).$$

[Remark: $\exp(A)$ is always invertible. Proof: A & $-A$ commute. so

$$\exp(A)\cdot\exp(-A)$$

$$= \exp(A-A)$$

$$= \exp(0) = I.$$

so $\exp(A)^{-1} = \exp(-A)$.

Analogue: $\exp(x) > 0 \ \forall\, x \in \mathbb{R}$]

Two Dynamical Systems:

- $\vec{x}'(t) = A\,\vec{x}(t)$

$$A = \frac{1}{6}\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix}$$

$$A = \left(\begin{array}{c|c} 4 & 1 \\ 1 & -2 \end{array}\right)\begin{pmatrix} 1 & \\ & -1/2 \end{pmatrix}\left(\begin{array}{c|c} 4 & 1 \\ 1 & -2 \end{array}\right)^{-1}$$

$\exp(At)$

$$= \left(\begin{array}{c|c} 4 & 1 \\ 1 & -2 \end{array}\right)\begin{pmatrix} e^{t} & \\ & e^{-t/2} \end{pmatrix}\left(\begin{array}{c|c} 4 & 1 \\ 1 & -2 \end{array}\right)^{-1}$$

Say $\vec{x}(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

Solution:

$$\vec{x}(t) = \exp(At)\cdot\vec{x}(0)$$

$$= \left(\begin{array}{c|c} 4 & 1 \\ 1 & -2 \end{array}\right)\begin{pmatrix} e^{t} & \\ & e^{-t/2} \end{pmatrix}\left(\begin{array}{c|c} 4 & 1 \\ 1 & -2 \end{array}\right)^{-1}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$= \quad \vdots$$

$$= \begin{pmatrix} 4e^{t} - 2e^{-t/2} \\ e^{t} + 4e^{-t/2} \end{pmatrix}$$

$$= 1 e^{t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} - 2 e^{-t/2} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Initial condition in terms of eigenvectors:

$$1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

Non-D'ble Example.

$$\vec{x}'(t) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \vec{x}(t)$$

Solution $\vec{x}(t) = \exp\left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} t\right) \vec{x}(0)$

Jordan Block.

$$\exp\left(\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} t\right)$$

$$= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2e^{\lambda t} & \cdots \\ & e^{\lambda t} & te^{\lambda t} & \ddots & \frac{1}{2}t^2e^{\lambda t} \\ & & & & te^{\lambda t} \\ & & & & e^{\lambda t} \end{pmatrix}$$

In our case

$$\exp\left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} t\right) = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}$$