HW5 Discussion:

Problem 1:

$$A = XBX^{-1}$$

$$\begin{aligned}\chi_A(\lambda) &= \det(\lambda I - A) \\ &= \det(\lambda XX^{-1} - XBX^{-1}) \\ &= \det(X(\lambda I - B)X^{-1}) \\ &= \det(\lambda I - B) \\ &= \chi_B(\lambda).\end{aligned}$$

Called conjugation, or "similarity" of matrices. Similar matrices have the same eigenvalues. [Not necessarily the same e.vectors.]

On the other hand, we can expand char poly true for any matrix.

$$\chi_A(\lambda) = \lambda^n - tr(A)\lambda^{n-1} + \cdots + (-1)^n \det(A)$$

$$\chi_B(\lambda) = \lambda^n - tr(B)\lambda^{n-1} + \cdots + (-1)^n \det(B).$$

Same polynomial $\Rightarrow$ same coeffs.

$$\Rightarrow \quad tr(A) = tr(B)$$

$$\det(A) = \det(B)$$ ← already knew this.

Factor $\chi_A(\lambda)$ over $\mathbb{C}$:

$$\chi_A(\lambda) = \boxed{(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)},$$ true for any matrix

$\lambda_1, \dots, \lambda_n \in \mathbb{C}$ not neces. distinct.

Expand:

$$\chi_A(\lambda) = \lambda^n - (\lambda_1 + \cdots + \lambda_n)\lambda^{n-1} + \cdots + (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n.$$

compare coeffs to get

$$tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Often quite useful.

In abstract linear alg. people don't want to choose a basis.

$$f : V \to V$$

with $\dim(V) = n$. Even if we don't think of f as a matrix, we can still define trace & det:

$$\operatorname{tr}(F) = \lambda_1 + \cdots + \lambda_n$$

$$\det(F) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Idempotent Matrices:

Suppose $P^2 = P$.

If $P\vec{x} = \lambda \vec{x}$, $\vec{x} \neq 0$.

$$\lambda \vec{x} = P\vec{x} = P^2 \vec{x} = \lambda^2 \vec{x}$$

$$(\lambda^2 - \lambda)\vec{x} = \vec{0}.$$

$\vec{x} \neq \vec{0} \implies \lambda^2 - \lambda = 0$

$\lambda(\lambda - 1) = 0$

$\lambda = 0$ or $\lambda = 1$.

Fact: $P^2 = P \implies P$ d'ble.

[There is now a proof in the notes.]

$\exists\ \vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^n$

$P\vec{x}_1 = 1\vec{x}_1 \quad \cdots \quad P\vec{x}_r = 1\vec{x}_r$

$P\vec{x}_{r+1} = 0\vec{x}_{r+1} \quad \cdots \quad P\vec{x}_n = 0\vec{x}_n$

$r = \text{rank}(P)$

$r = 0$ or $r = n$ allowed.

$P = 0 \qquad P = I$.

$$P = \underbrace{(\vec{x}_1 | \cdots | \vec{x}_n)}_{X} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \underbrace{(\vec{x}_1 | \cdots | \vec{x}_n)^{-1}}_{X^{-1}}$$

Write in Block form

$$P = \left(A \,\middle|\, *\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c} B \\ \hline * \end{array}\right)$$

A = first r cols of X $(n \times r)$

B = first r rows of X^{-1} $(r \times n)$.

$$= \left(A \,\middle|\, *\right)\left(\begin{array}{c} B \\ \hline 0 \end{array}\right) = AB.$$

$$P = \underset{n\times r}{A}\;\underset{r \times n.}{B}$$

Also:

$$X^{-1}X = I$$

$$\left(\begin{array}{c} B \\ \hline * \end{array}\right)\left(A \,\middle|\, *\right) = \left(\begin{array}{c|c} BA & * \\ \hline * & * \end{array}\right)$$

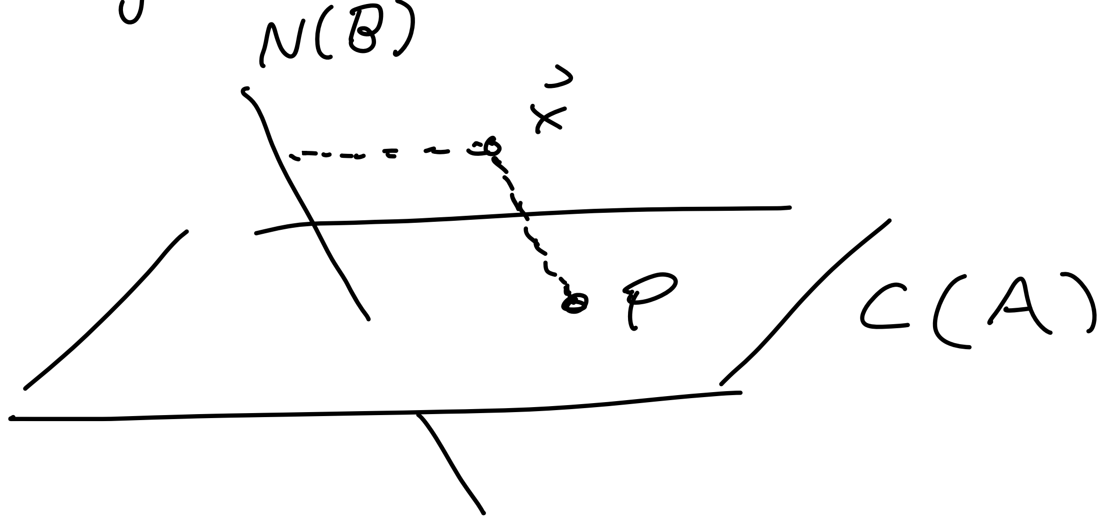
$$= \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & I_{n-r} \end{array}\right)$$

$$\Longrightarrow \underset{r\times n}{B}\,\underset{n\times r}{A} = I_r.$$

Summary:

$$P^2 = P \Longrightarrow P = AB$$
where $BA = I_r$.

Geometry:



$$\mathrm{Col}(P) = \mathrm{Col}(A\underset{\uparrow}{B}) = \mathrm{Col}(A)$$
ind rows.

$$B\vec{x} = \vec{0} \Longrightarrow P\vec{x} = AB\vec{x} = A\vec{0} = \vec{0}.$$

If $N(B) \perp C(A)$ then one can show $P = AB = A(A^TA)^{-1}A^T$.

Normal Matrices / Operators.

V inner product space over $\mathbb{C}$.

$A: V \to V$ "bounded" operator

$\Longrightarrow \exists\, A^*: V \to V$

$$\langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, A^*\vec{y}\rangle$$

$$\langle \vec{x}, A\vec{y}\rangle = \langle A^*, \vec{x}\rangle$$

$\forall\, \vec{x}, \vec{y} \in V$.

$V = \mathbb{C}^n$ then A^* is conj. transpose.

$V = \mathbb{R}^n$ then $A^* = A^T$ is transpose.

Suppose $A^*A = AA^*$

say A is "normal".

Goal: E.vectors corr to different e.values are orthogonal.

(a) $\langle A\vec{x}, A\vec{y}\rangle$

$= \langle \vec{x}, A^*A\vec{y}\rangle$

$= \langle x, AA^*y\rangle$

$= \langle A^*x, A^*y\rangle.$

(b) Put $y = x$:

$$\|A\vec{x}\|^2 = \langle Ax, Ax\rangle$$

$$= \langle A^*x, A^*x\rangle = \|A^*\vec{x}\|^2$$

Normal operators preserve distance.

$$A\vec{x} = 0 \iff \|A\vec{x}\| = 0$$

$$\iff \|A^*\vec{x}\| = 0$$

$$\iff A^*\vec{x} = 0.$$

same nullspace.

(c) For any $\lambda \in \mathbb{C}$, consider

$$B = \lambda I - A.$$

Have $B^* = \lambda^* I - A^*$

$$B^* B = (\lambda^* I - A^*)(\lambda I - A)$$
$$= \lambda^* \lambda I - \lambda^* A - \lambda A^* + A^* A.$$
$$= \lambda^* \lambda I - \lambda^* A - \lambda A^* + AA^*$$
$$= (\lambda I - A)(\lambda^* I - A^*)$$
$$= BB^*.$$

[More generally, $f(A)^* f(A)$
$= f(A) f(A)^*$ $\forall$ polynomials f]

Conclusion:

$$A\vec{x} = \lambda \vec{x} \iff (\lambda I - A)\vec{x} = \vec{0}$$
$$\iff B\vec{x} = \vec{0}$$
$$\iff B^* \vec{x} = 0 \qquad B^* B = BB^*$$
$$\iff (\lambda^* I - A^*)\vec{x} = \vec{0}$$
$$\iff A^* \vec{x} = \lambda^* \vec{x}.$$

So if $A^*A = AA^*$ then

A, A^* have same e.vectors,

conjugate e.values.

Finally, (d).

Suppose $A\vec{x} = \lambda\vec{x}$ $\quad \lambda \neq \mu$.

$A\vec{y} = \mu\vec{y}$

Then $\lambda\langle x, y\rangle$

$= \langle \lambda^* x, y\rangle$

$= \langle A^* x, y\rangle$

$= \langle x, Ay\rangle$

$= \langle x, \mu y\rangle = \mu\langle x, y\rangle$.

$\lambda\langle x, y\rangle = \mu\langle x, y\rangle$

$(\lambda - \mu)\langle x, y\rangle = 0$

$\lambda - \mu \neq 0 \implies \langle x, y\rangle = 0$.

Special Cases:

$$A^T = A \qquad A^T A = I$$
$$A^* = A \qquad A^* A = I$$

Spectral Theorem:

$A^* A = A A^*$

$\implies \exists$ orthonormal basis

of e.vectors,

$$A = U \Lambda U^*$$

$U^* U = I$, Λ diag.

Problem 2:

A real $n \times n$.

$\chi_A(x)$ real degree n.

$f(x) = b_0 + b_1 x + \cdots + b_n x^n$

$b_0, \dots, b_n$ real.

$\alpha \in \mathbb{C}$.

$$\overline{f(\alpha)} = \overline{b_0 + b_1\alpha + \cdots + b_n\alpha^n}$$

$$= \overline{b_0} + \overline{b_1}\,\overline{\alpha} + \cdots + \overline{b_n}(\overline{\alpha})^n$$

$$= b_0 + b_1\overline{\alpha} + \cdots + b_n(\overline{\alpha})^n$$

$$= f(\overline{\alpha})$$

$$f(\alpha) = 0 \iff f(\overline{\alpha}) = 0.$$

$$f(\alpha) = 0 \implies f(\overline{\alpha}) = \overline{f(\alpha)} = \overline{0} = 0.$$

$$f(\overline{\alpha}) = 0$$

$$\implies f(\alpha) = \overline{\overline{f(\alpha)}}$$

$$= \overline{f(\overline{\alpha})} = \overline{0} = 0. \checkmark$$

Non-real complex roots come in conjugate pairs.

Corollary:

A real $\Rightarrow$ $\chi_A(x)$ has real coeffs.

So non-real complex eigenvalues come in conjugate pairs.

If n is odd, $\chi_A(x)$ must have a real eigenvalue!

If not, it would have an even # of eigenvalues. $\times$

Complex E.vectors of real matrices.

A real

$A\vec{x} = \lambda\vec{x}$, $\lambda \in \mathbb{C}$ <u>not</u> real.

Then $\vec{x} \notin \mathbb{R}^n$.

otherwise: $\lambda\vec{x} = A\vec{x} \in \mathbb{R}^n$

$\Longrightarrow \lambda \in \mathbb{R}.$

$\lambda \in \mathbb{C} - \mathbb{R}, \quad A\vec{x} = \lambda\vec{x}$

$\Longrightarrow \vec{x} \in \mathbb{C}^n - \mathbb{R}^n.$

$\lambda = a + ib \qquad a, b \in \mathbb{R}$

$\vec{x} = \vec{u} + i\vec{v} \qquad \vec{u}, \vec{v} \in \mathbb{R}$

$$
\begin{aligned}
A\vec{u} + iA\vec{v} &= A(\vec{u} + i\vec{v}) \\
&= A\vec{x} \\
&= \lambda\vec{x} \\
&= (a+ib)(\vec{u}+iv) \\
&= (a\vec{u} - b\vec{v}) + i(b\vec{u} + a\vec{v}).
\end{aligned}
$$

$$
\Longrightarrow \begin{cases} A\vec{u} = a\vec{u} - b\vec{v} \\ A\vec{v} = b\vec{u} + a\vec{v}. \end{cases}
$$

$\vdots$

Punchline.

If real A is d'ble, then

$\exists$ real basis $X = (x_1 | \cdots | x_n)$

such that.

$$A = X \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_m & & \\ & & & \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} & \\ & & & & \ddots \\ & & & & & \begin{pmatrix} a_n & -b_n \\ b_n & a_n \end{pmatrix} \end{pmatrix} X^{-1}.$$

rotations & dilations.