Review For Exam 2:

Given matrix A $(m \times n)$:

$$C(A) = \{ \text{lin comb of cols of } A \}.$$
$$= \{ A\vec{x} : \text{for all } \vec{x} \in \mathbb{R}^n \}$$

$$R(A) = C(A^T)$$

$$N(A) = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \}$$

$$N(A^T).$$

Example: $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & -3 & -6 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theory:

$$\begin{pmatrix} 1 & -1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ -2 & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{pmatrix} A$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underbrace{E_5 E_4 E_3 E_2 E_1}_{E \text{ invertible}} A$$

$$R(A) = R(EA)$$

$$= R\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

$$C(A) \neq C(EA).$$

But pivot cols RREF

$\longrightarrow$ basis for $C(A)$.

pivot cols of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ are 1 & 2. So 1 & 2 cols of A are a basis for col space:

$$C(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

<u>Alternatively</u>:

$$RREF(A^T) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C(A) = R(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}.$$

Nullspaces?

$$A\vec{x} = \vec{0}$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{RREF} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + z = 0 \\ y + 2z = 0 \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$$N(A) = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\} = \text{line}$$

$$R(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\} = \text{plane.}$$

$$N(A) = R(A)^{\perp}$$

$\Big[$ $A\vec{x} = \vec{0} \iff \vec{r_i}^T \vec{x} = 0$ for all rows $\vec{r_i}^T$ of A

$\iff \vec{x} \in R(A)^{\perp}$ $\Big]$

$R(A)$ = plane $-x - 2y + z = 0$

$\quad$ = line $\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}^{\perp}$

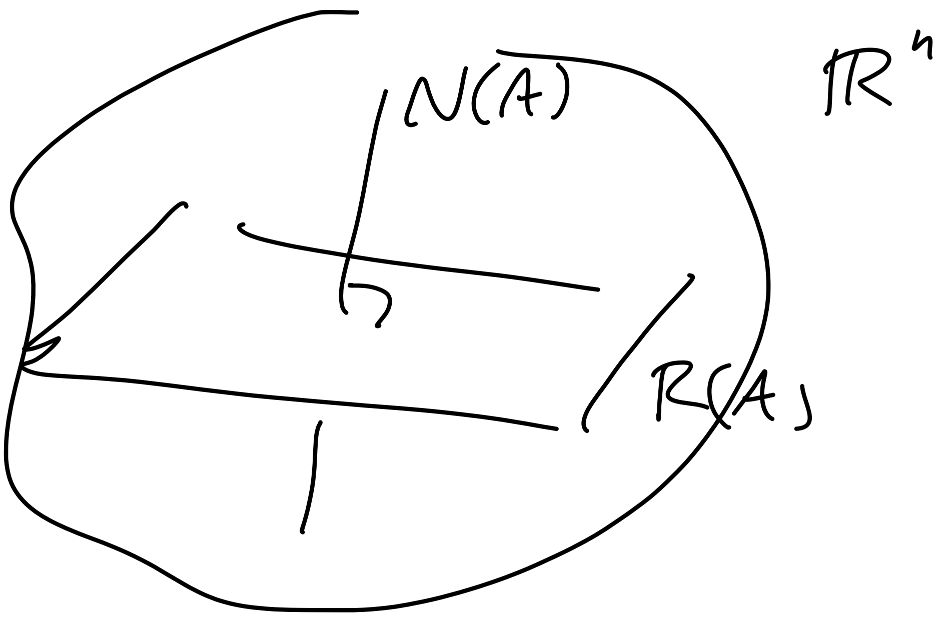
A $m \times n$.

$r = \dim R(A) = \dim C(A)$.

$\dim R(A) + \dim N(A) = n$ ✓

$\dim C(A) + \dim N(A^T) = m$.

$R(A),\ N(A) \subseteq \mathbb{R}^n$



$\dim N(A) = n - r$

$\dim N(A^T) = m - r.$

Existence of inverses.

A has left inverse $\iff r = n.$

$\iff \dim N(A) = 0$

$\vdots$

A has right inverse $\iff r = m$

A has two-sided inverse $\iff r = m = n.$

One-sided inverses NOT UNIQUE.

Two-sided inverse UNIQUE.

$$(A \mid I) \xrightarrow{\text{RREF}} (I \mid A^{-1})$$

$$E_k \cdots E_1 A = I$$

$$A = E_1^{-1} \cdots E_k^{-1}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -5 & -2 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2/5 & -1/5 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & 1/5 & 2/5 \\ 0 & 1 & 2/5 & -1/5 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix}$$

ALSO:

$$\begin{pmatrix} 1 & -2 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1/5 \end{pmatrix}\begin{pmatrix} 1 & \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & \\ -2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & \\ & -1/5 \end{pmatrix}^{-1}\begin{pmatrix} 1 & -2 \\ & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & -5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1/5 \end{pmatrix} \begin{pmatrix} 1 & \\ -2 & 1 \end{pmatrix}$$

Get the determinant:

$$\det \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$= \det \cancel{\begin{pmatrix} 1 & \\ 2 & 1 \end{pmatrix}} \det \cancel{\begin{pmatrix} 1 & \\ & -5 \end{pmatrix}} \det \cancel{\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}}$$

$$1 \qquad -5 \qquad 1.$$

$$\det \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = (1)(-1) - (2)\cdot(2)$$

$$= -5.$$

Least squares:

$$\begin{cases} x + y = 1 \\ x = 1 \\ 2x - y = 1 \end{cases}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

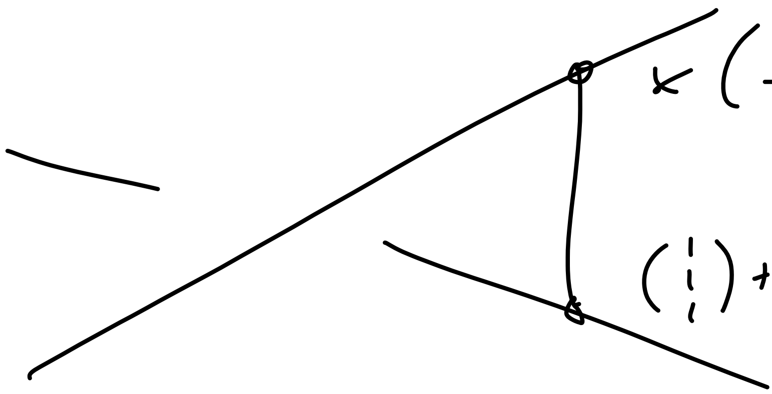
$$= \frac{1}{11} \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$= \frac{1}{11} \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

Maybe:

$$\underbrace{x \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

intersection of two lines in 3D.



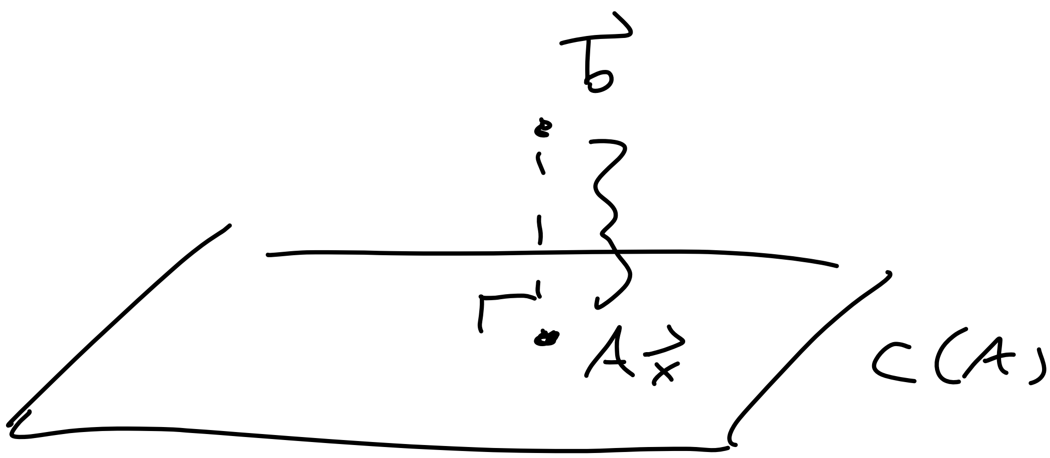
$$x\begin{pmatrix}1\\1\\2\end{pmatrix} = \frac{8}{11}\begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{4}{11}\begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

What did we do?

$$A\vec{x} = \vec{b}$$

IF $\vec{b} \notin C(A)$:



Minimize $\|A\vec{x} - \vec{b}\|$

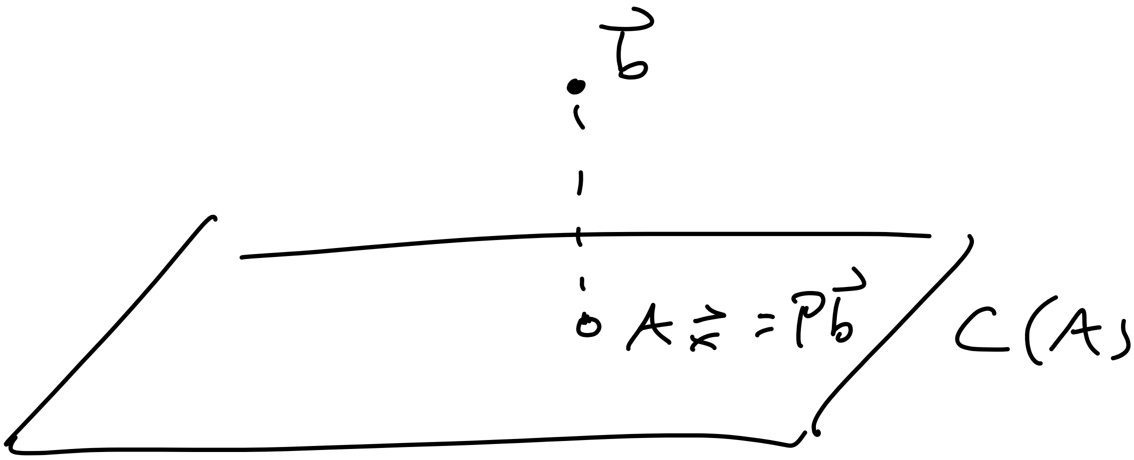
by taking $A\vec{x} - \vec{b} \perp C(A)$.

$$A^T(A\vec{x} - \vec{b}) \overset{!}{=} \vec{0}$$

$$A^T A\vec{x} - A^T\vec{b} = \vec{0}$$

$$A^T A\vec{x} = A^T\vec{b} \quad ✓$$

One more step. Let P be matrix that <u>projects</u> onto $C(A)$.



$$A^T A\vec{x} = A^T\vec{b}$$

If A has ind columns,

$$\vec{x} = (A^T A)^{-1}A^T\vec{b}$$

$$A\vec{x} = A(A^T A)^{-1}A^T\vec{b}$$

$$P\vec{b} = A(A^T A)^{-1}A^T\vec{b}$$

$$P = A(A^T A)^{-1}A^T$$

Plane $-x-2y+z=0$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$P = A(A^TA)^{-1}A^T.$$

OR: $Q = I - P$ projects onto the orthogonal subspace, i.e. the line span $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$.

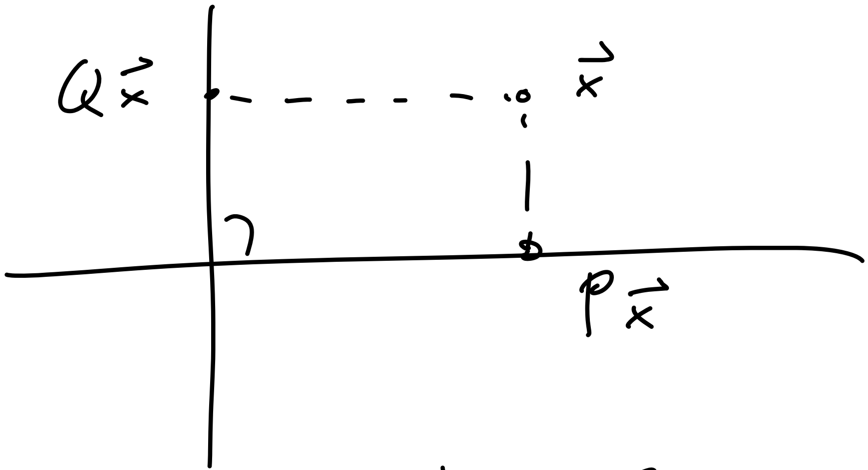
$$Q = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \Bigg[\underset{6}{\cancel{(-1\ -2\ 1)\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}}} \Bigg]^{-1} (-1\ -2\ 1)$$

$$= \frac{1}{6}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

$$P = I - Q = \cdots$$

$P^2 = P$ & $P^T = P$. $Q = I - P$

Then $Q^2 = Q$ & $Q^T = Q$.



$$\vec{x} = P\vec{x} + Q\vec{x}$$

$$(P\vec{x})^T (Q\vec{x}) = 0$$

Theory: $N(A^TA) = N(A)$.

$\vec{x} \in N(A) \Rightarrow A\vec{x} = \vec{0}$

$$\Rightarrow (A^TA)\vec{x} = A^T(A\vec{x})$$

$$= A^T\vec{0} = \vec{0}$$

$$\Rightarrow \vec{x} \in N(A^TA).$$

$\vec{x} \in N(A^TA) \Rightarrow A^TA\vec{x} = \vec{0}$.

TRICK:

$$\|A\vec{x}\|^2 = (A\vec{x})^T(A\vec{x})$$
$$= \vec{x}^T A^T A \vec{x}$$
$$= \vec{x}^T \vec{0} = 0.$$

$$\Longrightarrow \|A\vec{x}\| = 0 \Rightarrow A\vec{x} = 0$$
$$\Rightarrow \vec{x} \in N(A).$$

$(A^TA)^{-1}$ exists

$\Longleftrightarrow$ rank(A^TA)
$= \#$ columns (A^TA).

$\Longleftrightarrow$ rank(A)
$= \#$ columns (A)

$\Longleftrightarrow$ A has ind columns.

So for ind cols A,

$$A\vec{x} = \vec{b}$$
$$A^TA\vec{x} = A^T\vec{b}$$

$$\Longrightarrow \quad \vec{x} = (A^T A)^{-1} A^T \vec{b}$$

When A has ind cols,
least squares solution is UNIQUE.

Bilinear form $\varphi_B(\vec{x}, \vec{y}) = \vec{x}^T B \vec{y}$.

Important: $\varphi_B(\vec{e}_i, \vec{e}_j) = b_{ij}$

If $B = A^T A$
then $\varphi_B(\vec{x}, \vec{x}) \geq 0$.

If A has ind columns
then $\varphi_B(\vec{x}, \vec{x}) = 0 \Rightarrow \vec{x} = \vec{0}$.

We say φ is positive definite.
B is pos. def.

Why do we care?
φ is pos. def. then function

FORGET THIS.

$$f = b + \vec{b}^T \vec{x} + q(\vec{x}, \vec{x})$$

has a local min when $\vec{b} = \vec{0}$.

$$f = b + q(\vec{x}, \vec{x}) \geq b.$$

$$= b \iff \vec{x} = \vec{0}.$$

Properties of det:

$$\det(A^T) = \det(A)$$

$$\det(AB) = \det(A)\det(B).$$

Use it: A square,

$$\sqrt{\det(A^T A)} = |\det(A)|.$$

Also: Check AB invertible $\iff$ A & B invertible.

Recall M invertible $\iff \det(M) \neq 0$.

$\det(A) \neq 0$ $\det(B) \neq 0$

$\Rightarrow \det(AB) = \det(A)\det(B) \neq 0.$

Conversely $\det(AB) \neq 0$

$\Rightarrow \det(A)\det(B) \neq 0$

$\Rightarrow \det(A) \neq 0$ & $\det(B) \neq 0.$