Problem 5 on HW 5 will be optional. HW6 will replace Exam 3.

Applications:

Principal Axes Theorem (Old)

Descartes, Fermat (1600s)

Euler (1700)

High School:

$$f(x,y) = a + bx + cy + dx^2 + exy + fy^2.$$

Equation $f(x,y) = 0$ can be reduced to standard form by a translation & a rotation.

Standard forms:

parabola : $y = ax^2$ or $x = ay^2$

ellipse : $x^2/a^2 + y^2/b^2 = 1$

hyperbola : $\pm\left(\frac{x^2}{a^2} - y^2/b^2\right) = 1$.

Principal Axes Theorem:

$$f(x_1, \dots, x_n) = b + \sum b_i x_i + \sum b_{ij} x_i x_j$$

$$= b + \vec{b}^T \vec{x} + \vec{x}^T B \vec{x}$$

where $B^T = B$. If B^{-1} exists then we can find change of coords $\vec{u} = Q\vec{x} + \vec{t}$ where

$$Q^T Q = I \quad \text{(orthogonal)}$$

(generalized rotation)

such that

$$f(\vec{u}) = f(u_1, u_2, \dots, u_n)$$

$$= c + \lambda_1 u_1^2 + \lambda_2 u_2^2 + \dots + \lambda_n u_n^2$$

Proof: Since $B^T = B$, Spec. Thm. $\Rightarrow \exists\ Q^T Q = I$ so

$$B = Q \Lambda Q^T$$

with $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$.

Consider $\vec{u} = Q\vec{x} + \vec{t}$ for some vector $\vec{t}$, yet to be chosen. Then

$$f(\vec{u}) = F(Q\vec{x} + \vec{t})$$
$$= b + \vec{b}^T(Q\vec{x} + \vec{t}) + (Q\vec{x} + \vec{t})^T B (Q\vec{x} + \vec{t})$$
$$= b + \vec{b}^T Q\vec{x} + \vec{b}^T\vec{t} + \underbrace{(\vec{x}^T Q^T + \vec{t}^T) B (Q\vec{x} + \vec{t})}_{\vec{x}^T Q^T B Q \vec{x} + \vec{x}^T Q^T B \vec{t} + \vec{t}^T B Q \vec{x} + \vec{t}^T B \vec{t}}$$

$$\vec{x}^T Q^T B \vec{t} = \vec{t}^T B Q \vec{x}$$

[$B^T = B$ & $\vec{x}^T Q^T B \vec{t}$ scalar, so

$$\vec{x}^T Q^T B \vec{t} = (\vec{x}^T Q^T B \vec{t})^T$$
$$= \vec{t}^T B^T Q \vec{x}$$
$$= \vec{t}^T B Q \vec{x}. \quad]$$

$$= b + \vec{b}^T \vec{t} + \vec{t}^T B \vec{t} + \vec{b}^T Q \vec{x} + 2\vec{t}^T B Q \vec{x} + \vec{x}^T \underbrace{\cancel{Q^T B} Q}_{\Lambda} \vec{x}.$$

$$= \left(b + \vec{b}^T \vec{t} + \vec{t}^T B \vec{t}\right) + \underbrace{\left(\vec{b}^T Q + 2\vec{t} B Q\right)}_{=0.} \vec{x} + \vec{x}^T \Lambda \vec{x}.$$

$$\vec{b}^T Q + 2\vec{t} B Q = \vec{0}^T$$

$$2\vec{t} B Q = -\vec{b}^T Q$$

$$\vec{t} B Q = -\frac{1}{2}\vec{b}^T Q$$

assume exists

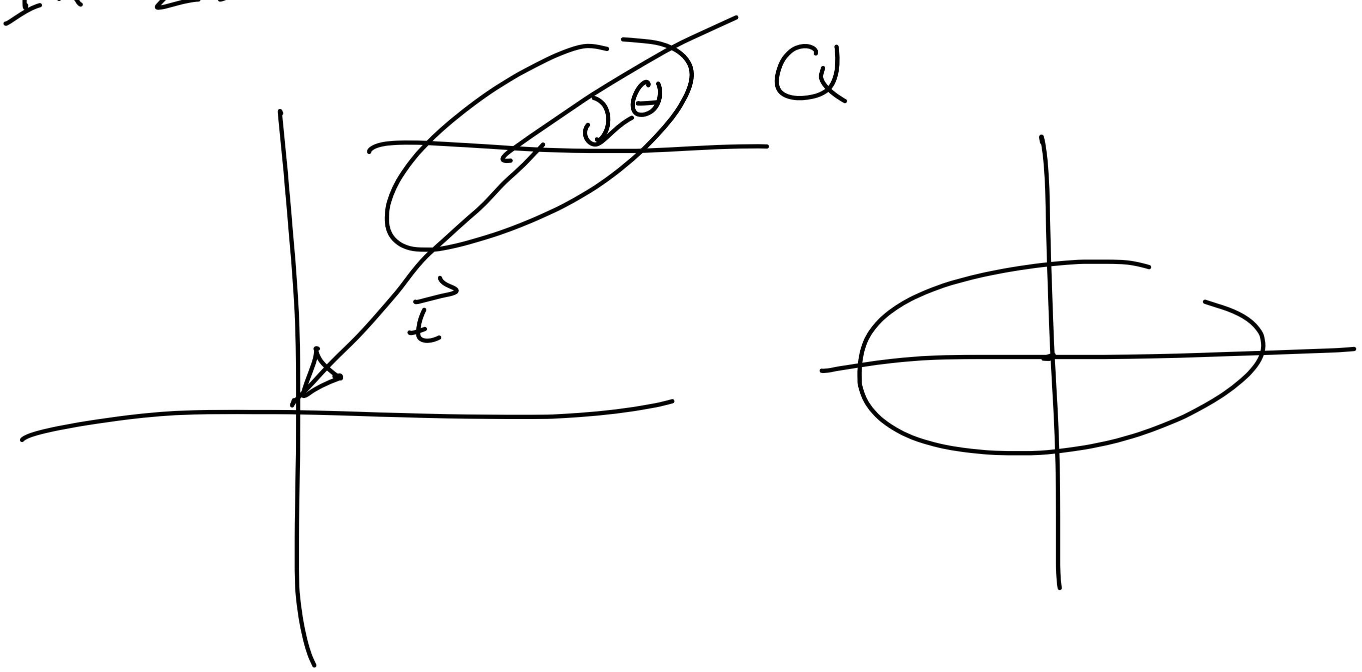
$$\vec{t} = -\frac{1}{2}\vec{b}^T \cancel{Q Q^{-1}} B^{-1}$$

$$\vec{t} = -\frac{1}{2}\vec{b}^T B^{-1}$$

Finally (oops):

$$f(u_1, \dots, u_n) = c + \lambda_1 x_1^2 + \dots + \lambda_n x_n^2$$

In 2D:



Apply to $F(x,y) = 2x^2 + 2xy + y^2 + x + 1$.

$$= 1 + \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} + \begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}2 & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}.$$

$$R_\theta^{-1}\begin{pmatrix}2 & 1 \\ 1 & 1\end{pmatrix}R_\theta = \begin{pmatrix}\lambda & \\ & \mu\end{pmatrix}.$$

||

$$\begin{pmatrix}2c^2 + 2sc + 1s^2 & \cdots \\ (1-2)sc + 1(c^2 - s^2) & \cdots\end{pmatrix}.$$

$c = \cos\theta$
$s = \sin\theta$

Want $(1-2)sc + (c^2 - s^2) = 0$.

$\rightsquigarrow \tan\theta = 1/1 = 1$

$$\theta = \frac{\pi}{4}.$$

Formula: To diagonalize symmetric

$$A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

Rotate by θ where

$$\tan\theta = b/(a-c).$$

Translation ... I.O.U.

3D Version (Euler's Rotation Theorem): You can still do it with a rotation.

<u>Theorem</u>: Any 3×3 $Q^TQ = I$ can be written as

$$Q = X \left(\begin{array}{c|cc} \pm 1 & 0 & 0 \\ \hline 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{array}\right) X^{-1}$$

IF $\det(Q) = -1$ pick -1

$\det(Q) = 1$ pick $+1$

Implies any quadratic

$$F(x, y, z)$$

Can be diagonalized by a rotation.

Positive Definite Matrices.

We previously showed,

for $B^T = B$:

$$\vec{x}^T B \vec{x} \geq 0 \quad \forall \vec{x} \Longleftarrow B = A^T A$$

If A has ind cols then

$\vec{x}^T B \vec{x} = 0 \iff \vec{x} = 0.$

But it's really hard to show

B semidef $\Longrightarrow B = A^T A$. It's

like finding the "square root"

of a matrix.

Spectral Theorem:

$B^T = B \Longrightarrow B = Q \Lambda Q^T$

for some $Q^T Q = I$.

Furthermore B pos semidef

$\Longrightarrow$ e.values $\lambda \geq 0$.

[we know $B^T = B \Rightarrow$ real e.values.

I.O.U. $\vec{x}^T B \vec{x} \geq 0 \ \forall \vec{x} \Rightarrow \geq 0$ e.values]

$$B = Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} Q^T$$

TRICK: Λ is diagonal with entries ≥ 0 so it makes sense to take the "square root":

$$\sqrt{\Lambda} \stackrel{\text{DEF}}{=} \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}$$

Check: $\sqrt{\Lambda} \cdot \sqrt{\Lambda} = \Lambda$

In this special case we can lift this to a "square root" of B. Take

$$A^T = Q\sqrt{\Lambda}$$

$$A = \sqrt{\Lambda}^{\,T} Q^T = \sqrt{\Lambda}\, Q^T$$

Then

$$A^T A = Q\sqrt{\Lambda}\sqrt{\Lambda}\, Q$$
$$= Q \Lambda Q^T = B \quad \checkmark$$

Not unique, but this is an algorithm to factor semi-def matrix

B semidef $\rightsquigarrow$ $B = A^T A$.