Instead of in-class exam, just have HW 6 as a final project.

Gram-Schmit Orthogonalization:

[HW5. Problem 6 (optional)]

Let V be an inner product space. Given any basis $\vec{a}_1, \vec{a}_2, \ldots \in V$

[Works also for Hilbert spaces.]

we can create an orthogonal basis,

$$b_1, b_2, \ldots \in V.$$

which can then be made orthonormal. Furthermore, we have property

$$\text{span}\{\vec{b}_1, \ldots, \vec{b}_k\} = \text{span}\{\vec{a}_1, \ldots, \vec{a}_k\}$$

Gram-Schmit Procedure:

- $\vec{b}_1 := \vec{a}_1$
- $\vec{b}_2 := a_2 - P_1(\vec{a}_2)$

where $P_1 =$ proj onto line $\vec{b}_1$.

$$\vec{b}_2 = a_2 - \frac{\langle \vec{a}_2, \vec{b}_1 \rangle}{\langle \vec{b}_1, \vec{b}_1 \rangle} \vec{b}_1$$

- $\vec{b}_3 = \vec{a}_3 - P_2(\vec{a}_3)$

where $P_2(\vec{a}_3) = \frac{\langle \vec{a}_3, \vec{b}_1 \rangle}{\langle \vec{b}_1, \vec{b}_1 \rangle} \vec{b}_1 + \frac{\langle \vec{a}_3, \vec{b}_2 \rangle}{\langle \vec{b}_2, \vec{b}_2 \rangle} \vec{b}_2$

$\vdots$

- $\vec{a}_{k+1} = \vec{a}_k - P_k(\vec{a}_k)$

$$= \vec{a}_k - \sum_{i=1}^{k} \frac{\langle \vec{a}_k, \vec{b}_i \rangle}{\langle \vec{b}_i, \vec{b}_i \rangle} \vec{b}_i$$

HW5.6: Check $\langle \vec{b}_i, \vec{b}_j \rangle = 0$, $i \neq j$.

And $\operatorname{span}\{\vec{b}_1, \dots, \vec{b}_k\} = \operatorname{span}\{\vec{a}_1, \dots, \vec{a}_k\}$.

Example in $L^2[-1,1]$.

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x) g(x)\, dx$$

Apply Gram-Schmidt to the "obvious basis" $1, x, x^2, x^3, x^4, \ldots$ to obtain the "Legendre polynomials".

$$f_0(x),\ f_1(x),\ f_2(x),\ \ldots$$

The matrix form of Gram-Schmidt is called "QR Factorization" [Complex "UT Factorization"].

Let $\vec{a}_1, \ldots, \vec{a}_n$ be basis for $\mathbb{C}^n$. Then G-S basis $\vec{b}_1, \ldots, \vec{b}_2$ satisfies

$$\vec{a}_1 = \vec{b}_1$$

$$\vec{a}_2 = \frac{\langle \vec{a}_2, \vec{b}_1 \rangle}{\langle \vec{b}_1, \vec{b}_1 \rangle} \vec{b}_1 + \vec{b}_2$$

$\vdots$

$$\vec{a}_k = \sum_{i=1}^{k-1} \frac{\langle \vec{a}_k, \vec{b}_i \rangle}{\langle \vec{b}_i, \vec{b}_i \rangle} b_i + \vec{b}_k .$$

Matrices:

$$\left(\vec{a}_1 \middle| \cdots \middle| \vec{a}_n \right) = \left(\vec{b}_1 \middle| \cdots \middle| \vec{b}_n \right) \begin{pmatrix} 1 & \frac{\langle a_2, b_1 \rangle}{\langle b_1, b_1 \rangle} & \cdots & \frac{\langle a_n, b_1 \rangle}{\langle b_1, b_1 \rangle} \\ & 1 & & \\ & & \ddots & \frac{\langle a_n, b_{n-1} \rangle}{\langle b_{n-1}, b_{n-1} \rangle} \\ 0 & & & 1 \end{pmatrix}$$

$$A = BR$$

Observe: $A_k = \left(\vec{a}_1 \middle| \cdots \middle| \vec{a}_k \right)$

$B_k = \left(\vec{b}_1 \middle| \cdots \middle| \vec{b}_k \right)$

R_k = 1st k rows of R.

$A_k, B_k \quad n \times k$

$R_k \quad k \times n$.

Then $A_k = B_k R_k$

$$C(A_k) = C(B_k R_k) = C(B_k)$$

ind rows.

By construction, B has orthogonal columns $\vec{b}_1, \dots, \vec{b}_n \in \mathbb{C}$. Turn B into unitary by scaling its columns:

$$U = B \begin{pmatrix} 1/\langle b_1, b_1 \rangle & & \\ & \ddots & \\ & & 1/\langle b_n, b_n \rangle \end{pmatrix}$$

$$\Downarrow$$

$$A = UT$$

$$= U \begin{pmatrix} \langle a_1 b_1 \rangle & \langle a_2 b_1 \rangle & \cdots & \langle a_n b_1 \rangle \\ & \langle a_2 b_2 \rangle & & \vdots \\ & 0 & \ddots & \\ & & & \langle a_n b_n \rangle \end{pmatrix}$$

If A is real, get

$$A = QR$$

real orthog. $Q^T = Q^{-1}$

upper triangular.

How to compute eigenvalues:

① Factor char poly $\chi_A(\lambda)$.

Hard non-linear problem!

② QR Algorithm.

Idea: $A = XBX^{-1}$ A, B are conjugate.

then A & B have the same eigenvalues [HW 6.1].

$$\det(\lambda I - XBX^{-1})$$

$$= \det(\lambda X X^{-1} - X B X^{-1})$$

$$= \det(X(\lambda I - B) X^{-1})$$

$$= \det(\lambda I - B).$$

Recall Schur's Theorem:

$$A = U T U^{-1}$$

T is upper triangular, $U^{-1} = U^*$.

Eigenvalues of A

= e.values of T

= diag entries of T.

$$\begin{pmatrix} t_{11} & & * \\ & \ddots & \\ 0 & & t_{nn} \end{pmatrix}$$ has e.values $t_{11}, t_{22}, \ldots, t_{nn}$.

Problem: Proof of Schur's Thm assumed that we already <u>know</u> the e.values.

The QR-algorithm gives a way to recursively approximate Schur decomp, hence approximate the eigenvalues.

Start with matrix A.

apply Gram-Schmidt

$$T_1 := A.$$

$$T_1 = Q_1 R_1 \qquad (R_1 = Q_1^{-1} T_1)$$

$$T_2 := R_1 Q_1 \qquad \text{(weird!)}$$

$$= Q_1^{-1} T_1 Q_1$$

$$= Q_1^{-1} A Q_1$$

$$T_3 = R_2 Q_2 \quad \text{for } T_2 = Q_2 R_2$$

$$= Q_2^{-1} T_2 Q_2$$

$$= Q_2^{-1} Q_1^{-1} A Q_1 Q_2.$$

Continue :

$$T_k = Q_k^{-1} \cdots Q_2^{-1} Q_1^{-1} \, A \, Q_1 Q_2 \cdots Q_k .$$

converges to Schur's Theorem.

$$T = Q^{-1} A Q .$$

or $A = Q T Q^{-1}$.

[Q real orth : $Q^{-1} = Q^T$].

T is upper triang.

eigenvalues of A

= eigenvalues of T

= diag. entries of T.

Recall the Spectral Theorem.

If $A^*A = AA^*$, then A has an o.n. basis of e. vectors.

$$A = U \Lambda U^*$$

where $U^* = U^{-1}$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

= diagonal matrix of eigenvalues.

If A is <u>real</u>, we can take U to be real. $Q = U$.

$$Q^* = Q^T$$

$Q^{-1} = Q^T$ is orthogonal.

First Application:
(Principal Axes Theorem)

Consider real $A^T = A$.
e.values are <u>real</u>, so
$\exists$ orthogonal $Q^T = Q^{-1}$

$$A = Q \Lambda Q^{-1}$$

Λ is real, diagonal.

e.g. degree 2 polynomial

$$f(x,y) = b + \vec{b}^T \vec{x} + \vec{x}^T B \vec{x}$$

with B symmetric.

Claim: $f(x,y) = 0$ is a
conic section: parab, ellipse, hyperb.

It's just been rotated & translated away from standard position.

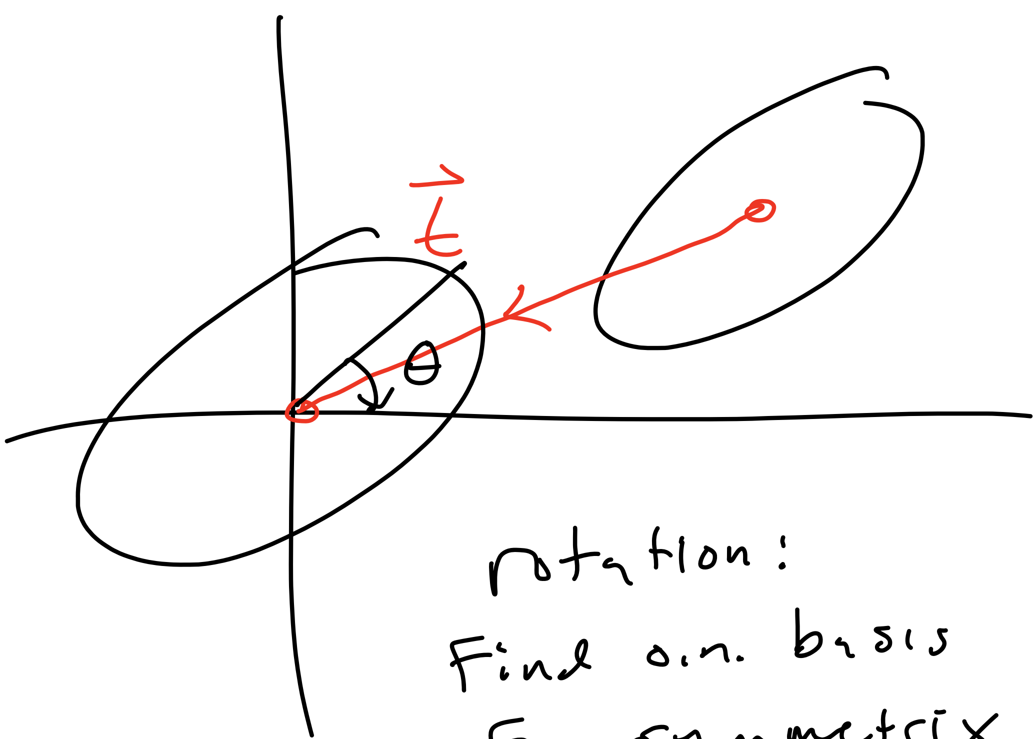
<u>Proof</u>: (1) translate to the origin:

$$F(\vec{x}+\vec{t})$$

$$= b + \vec{b}^T(\vec{x}+\vec{t}) + (\vec{x}+\vec{t})^T B (\vec{x}+\vec{t}).$$

$$= b + (\underbrace{\vec{b}^T + 2\vec{x}^T B\vec{t}}_{})\vec{x} + \cdots$$

To be continued...



rotation:
Find o.n. basis
for symmetrix
2x2 matrix.