Today: The Spectral Theorem.
(Principal Axes Theorem).

Recall:

λ e.vector $\iff \exists \vec{x} \neq \vec{0},\ A\vec{x} = \lambda \vec{x}$

$\iff (\lambda I - A)\vec{x} = \vec{0},\quad \vec{x} \neq \vec{0}$

$\iff N(\lambda I - A)$ not $\{\vec{0}\}$

$\iff \det(\lambda I - A) = 0.$

i.e. e.values are the roots of the characteristic polynomial

$$\chi_A(x) = \det(xI - A).$$

For any polynomial $f(x)$ we can evaluate at A:

$$f(x) = b_0 + b_1 x + \cdots + b_k x^k$$

$$f(A) = b_0 I + b_1 A + \cdots + b_k A^k.$$

Cayley - Hamilton Theorem:

Any square matrix A is a "root" of its own char poly:

$$\chi_A(A) = \text{zero matrix}.$$

Proof: Assume A diag'ble:

$$A = X \Lambda X^{-1}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

By definition, every e.value of A is a root of χ_A, so

$$\chi_A(\Lambda) = \begin{pmatrix} \cancel{\chi_A(\lambda_1)}\ 0 & & \\ & \ddots & \\ & & \cancel{\chi_A(\lambda_n)}\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} = 0.$$

But then,

important.

$$\chi_A(A) = \chi_A(X\Lambda X^{-1})$$

$$= X \cdot \chi_A(\Lambda) \cdot X^{-1}$$

$$= X \cdot 0 \cdot X^{-1} = 0.$$

[$A = XBX^{-1}$. For any poly $f(x)$,

$$f(A) = X \cdot f(B) \cdot X^{-1}.$$

Recall:

$$A^k = (XBX^{-1})^k$$

$$= (XBX^{-1})(XBX^{-1})^{k-1}$$

$$= XB\cancel{X^{-1}X}B^{k-1}X^{-1}$$

$$= XB^kX^{-1}.$$]

Cayley-Hamilton follows for non-diag'ble matrices "by continuity" $\square$

If χ_A has distinct roots
we showed A has a basis of
e.vectors so A is d'ble.

Not "if and only if".

e.g. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has

$\chi_A(x) = (x-1)^2$ repeated e.value.

But it's still d'ble.

<u>Better Theorem</u> (Won't prove it).

A is d'ble $\iff$

$f(A) = 0$ for <u>some</u> polynomial
with distinct roots.

e.g. $P^2 = P \implies P$ db'le.

because $f(x) = x^2 - x = x(x-1)$
has distinct roots.

e.g. $R^n = I \implies R$ d'ble.

because $f(x) = x^n - 1$ has distinct roots:

$$f(x) = \prod_{k=1}^{n} \left(x - e^{2\pi i k/n}\right).$$

Simplest Non-Example:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$\chi_A(x) = (x-1)^2 \quad \text{repeated root.}$$

But A is <u>not</u> a root of any smaller polynomial.

$$f(x) = x - 1.$$

$$f(A) = A - I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

[Remark: Nilpotent matrices, i.e.

$A^k = 0$ for some k,

are the abstruction to d'zation.]

The Spectral Theorem
(Principal Axes Theorem).

Call a matrix A <u>normal</u> when

$$\boxed{A^*A = AA^*}$$

$$A^TA = AA^T.$$

Includes several important classes of matrices:

$$A^T = A$$
$$A^* = A$$
$$A^TA = I$$
$$A^*A = I.$$

Theorem: Not only are these diagonalizable, they are unitarily diagonalizable.

Meaning they have o.n. basis

of e.vectors.

$$A^*A = AA^*$$

$\Longrightarrow \exists$ e.vectors $\vec{u}_1, \dots, \vec{u}_n \in \mathbb{C}^n$

such that $\vec{u}_i^* \vec{u}_j = \delta_{ij}$

Implies $U = (\vec{u}_1 | \cdots | \vec{u}_n)$ is

a unitary matrix:

$$U^*U = \begin{pmatrix} \vec{u}_1^* \\ \hline \vdots \\ \hline \vec{u}_n^* \end{pmatrix} (\vec{u}_1 | \cdots | \vec{u}_n)$$

$$= \begin{pmatrix} \vec{u}_1^* \vec{u}_1 & \cdots & \vec{u}_1^* \vec{u}_n \\ \vdots & & \vdots \\ \vec{u}_n^* \vec{u}_1 & \cdots & \vec{u}_n^* u_n \end{pmatrix} = I.$$

Hence

$$A = U \Lambda U^{-1}$$

$$A = U \Lambda U^*.$$

Proof uses an auxiliary result called Schur's Theorem.

Any square matrix is unitarily triangularizable:

$$A = U T U^*$$

where $U^* U = I$ &

$$T = \begin{pmatrix} t_{11} & & & * \\ & t_{22} & & \\ & & \ddots & \\ 0 & & & t_{nn} \end{pmatrix}.$$

Assume this for the moment. Then Spectral Theorem follows right away:

e.g. Take $A^T = A$ symmetric and <u>real</u>. Then

Schur: $A = Q T Q^T$

where $Q^T Q = I$ is <u>real</u>

orthogonal matrix.

Equivalently: $T = Q^T A Q$.

Use the fact that $A^T = A$ to see

$$T^T = (Q^T A Q)^T$$

$$= Q^T A^T Q^{TT}$$

$$= Q^T A Q = T.$$

Have $T = \begin{pmatrix} t_{11} & & * \\ & \ddots & \\ 0 & & t_{nn} \end{pmatrix}$

& $T = T^T = \begin{pmatrix} t_{11} & & 0 \\ & \ddots & \\ * & & t_{nn} \end{pmatrix}$.

$\Longrightarrow$ T is diagonal:

$$T = \begin{pmatrix} t_{11} & & 0 \\ & \ddots & \\ 0 & & t_{nn} \end{pmatrix}.$$

Proof of Schur:

Square matrix A.

Any matrix has at least one eigenvalue $\in \mathbb{C}$. Let

char poly has at least one root in $\mathbb{C}$.

$$A\vec{u}_1 = t_{11}\vec{u}_1$$

For some $\vec{u}_1 \in \mathbb{C}^n$, $t_{11} \in \mathbb{C}$.

Assume $\|\vec{u}_1\| = 1$. Choose any unitary matrix with 1st col $\vec{u}_1$:

$$\mathcal{U}_1 = \left(\vec{u}_1 \,\middle|\, \underbrace{\vec{u}_2 \,\middle|\, \cdots \,\middle|\, \vec{u}_n}_{\text{easy to choose.}} \right).$$

So $\mathcal{U}_1^* = \mathcal{U}_1^{-1}$. Then

$\mathcal{U}_1^* A\, \mathcal{U}_1$

$$= \begin{pmatrix} \vec{u}_1^* \\ \hline \vdots \\ \hline \vec{u}_n^* \end{pmatrix} \left(A\vec{u}_1 \,\middle|\, A\vec{u}_2 \,\middle|\, \cdots \,\middle|\, A\vec{u}_n \right)$$

$$= \begin{pmatrix} \vec{u}_1^{\,*} \\ \hline \vdots \\ \hline \vec{u}_n^{\,*} \end{pmatrix} \left(t_{11}\vec{u}_1 \;\middle|\; * \;\middle|\; \cdots \;\middle|\; * \right).$$

$$= \left(\begin{array}{c|ccc} t_{11} & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{array} \right)$$

Now use induction.

$\exists$ unitary $\mathcal{U}_2$, $\mathcal{U}_2^* A_2 \mathcal{U}_2 = T_2$

where T_2 is upper triangular.

Finally, let

$$\mathcal{U} = \mathcal{U}_1 \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathcal{U}_2 & \\ 0 & & & \end{array} \right),$$

which satisfies $\mathcal{U}^*\mathcal{U} = I$:

$$\left(\begin{array}{c|c} 1 & \\ \hline & \mathcal{U}_2^* \end{array} \right) \cancel{\mathcal{U}_1^* \mathcal{U}_1} \left(\begin{array}{c|c} 1 & \\ \hline & \mathcal{U}_2 \end{array} \right)$$

$$= \left(\begin{array}{c|c} 1 & \\ \hline & \mathcal{U}_2^* \mathcal{U}_2 \end{array}\right)$$

$$= \left(\begin{array}{c|c} 1 & \\ \hline & I_{n-1} \end{array}\right) = I_n \quad \checkmark$$

$\mathcal{U}^* A \,\mathcal{U}$

$$= \left(\begin{array}{c|c} 1 & \\ \hline & \mathcal{U}_2^* \end{array}\right) \mathcal{U}_1^* A\, \mathcal{U}_1 \left(\begin{array}{c|c} 1 & \\ \hline & \mathcal{U}_2 \end{array}\right)$$

$$= \left(\begin{array}{c|c} 1 & \\ \hline & \mathcal{U}_2 \end{array}\right) \left(\begin{array}{c|c} t_{11} & * \cdots * \\ \hline 0 & \\ \vdots & A_2 \\ 0 & \end{array}\right) \left(\begin{array}{c|c} 1 & \\ \hline & \mathcal{U}_2 \end{array}\right)$$

$$= \left(\begin{array}{c|c} t_{11} & * \cdots * \\ \hline 0 & \\ \vdots & \mathcal{U}_2^* A_2 \mathcal{U}_2 \\ 0 & \end{array}\right)$$

$$= \left(\begin{array}{c|c} t_{11} & * \cdots * \\ \hline 0 & \\ \vdots & T_2 \\ 0 & \end{array}\right) \quad \text{triangular} \ \checkmark$$