Plan: HW5 due Mon Nov 21.
HW6 due Fri Dec 2.
Exam 3 Wed Dec 7.

Eigenvalues & Eigenvectors.
("Spectral Theory")

Recall: λ is an e.value of square matrix A:

$$\exists\, \vec{x} \neq \vec{0}, \quad A\vec{x} = \lambda \vec{x}.$$

Given any poly $f(x) = b_0 + b_1 x + \cdots + b_k x^k$
we define the matrix

$$f(A) = b_0 I + b_1 A + \cdots + b_k A^k.$$

Then

$$A\vec{x} = \lambda \vec{x} \implies f(A)\vec{x} = f(\lambda)\vec{x}.$$

no content

Proof: $f(A)\vec{x}$

$= \left(\sum b_i A^i\right)\vec{x}$

$= \sum b_i (A^i \vec{x})$

$= \sum b_i \lambda^i \vec{x}$

$= f(\lambda)\vec{x}.$

If $\vec{x} \neq \vec{0}$, $A\vec{x} = \lambda\vec{x}$, $f(A) = 0$

then $f(\lambda) = 0$.

Proof: $\vec{0} = 0\vec{x}$

$= f(A)\vec{x}$

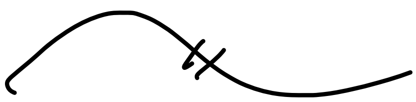
$= f(\lambda)\vec{x}.$

Since $\vec{x} \neq \vec{0}$, $f(\lambda) = 0$.

Who cares?

If $f(A) = 0$ then every

eigenvalue λ of A satisfies the equation $f(\lambda) = 0$.



Applications:

- If $P^2 = P$

  then $P^2 - P = 0$

so any e.value λ of P satisfies

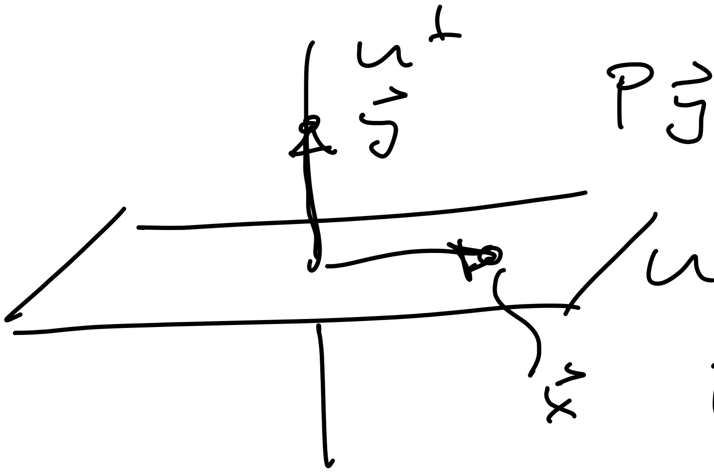
$$\lambda^2 - \lambda = 0$$

$$\lambda(\lambda - 1) = 0$$

$$\lambda = 0 \text{ or } 1.$$

e.g. projection (also has $P^T = P$).

onto subspace $U \subseteq \mathbb{R}^n$.



$P\vec{y} = \vec{0} = 0\vec{y}$.

$P\vec{x} = 1\vec{x}$.

Then U = 1-eigenspace
$U^\perp$ = 0-eigenspace.

- $F^2 = I$

$F^2 - I = 0$

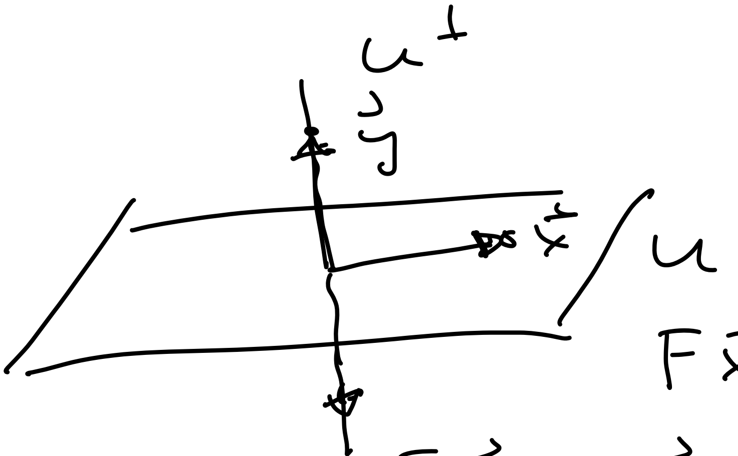
Then e.values λ satisfy

$\lambda^2 - 1 = 0$

$(\lambda - 1)(\lambda + 1) = 0$

$\lambda = +1$ or -1.

e.g. <u>reflection</u> across subspace $U \leq \mathbb{R}^n$.



$F\vec{x} = \vec{x} = 1\vec{x}$

$F\vec{y} = -\vec{y} = -1\vec{y}$.

U = 1-eigenspace

$U^\perp$ = (-1)-eigenspace.

- $R^n = I$

$$R^n - I = 0$$

The e.values λ satisfy

$$\lambda^n - 1 = 0.$$

$$\lambda^n = 1$$

$$\lambda = e^{2\pi i k/n} \quad \text{For } k \in \mathbb{Z}.$$

not real
unless $k = n$.
or n even & $k = n/2$.

$$e^{2\pi i} = 1 \qquad e^{2\pi i/2} = -1.$$

e.g. rotation matrices R.
Rotate by angle $2\pi/n$ around
some codim 2 subspace
then $R^n = I$.

e.g. $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$

has eigenvalues $e^{\pm i\theta}$.

Extreme Case:

Let $f(x) = \chi_A(x) = \det(xI - A)$.

The Cayley-Hamilton Theorem says $f(A) = 0$.

2×2.

$$\chi_A(x) = x^2 - (a+d)x + (ad-bc)$$

$$\chi_A(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \cdots\cdots = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Mysterious!

$\chi_A(A) = 0 \implies \chi_A(\lambda) = 0$

for all e.values. Well, duh!

Bring in A^T & A^*.

Let A be unitary $A^*A = I$

(includes orthogonal $A^TA = I$).

Then e.values λ satisfy $|\lambda| = 1$.

<u>Proof</u>: Suppose $A\vec{x} = \lambda\vec{x}$, $\vec{x} \neq \vec{0}$.

Note:

$$\|A\vec{x}\|^2 = \langle A\vec{x}, A\vec{x}\rangle$$
$$= \langle \vec{x}, A^*A\vec{x}\rangle.$$
$$= \langle \vec{x}, I\vec{x}\rangle$$
$$= \langle \vec{x}, \vec{x}\rangle = \|\vec{x}\|^2$$

[Unitary (orth.) matrices preserve lengths!]

Then

$$\|\vec{x}\|^2 = \|A\vec{x}\|^2$$
$$= \|\lambda\vec{x}\|^2 = |\lambda|\,\|\vec{x}\|^2$$

since $\vec{x} \neq \vec{0} \implies \|\vec{x}\| \neq 0$

$$\implies |\lambda| = 1.$$

Symmetric $A^T = A$

Hermitian $A^* = A$.

(self-adjoint)

Claim: Eigenvalues are <u>real</u>.

<u>Proof</u>: Let $A\vec{x} = \lambda\vec{x}$, $\vec{x} \neq \vec{0}$.

$$\lambda\|\vec{x}\|^2 = \lambda\langle\vec{x},\vec{x}\rangle$$
$$= \langle\vec{x},\lambda\vec{x}\rangle$$
$$= \langle\vec{x},A\vec{x}\rangle$$

$$= \langle A^* \vec{x}, \vec{x} \rangle$$

$$= \langle A \vec{x}, \vec{x} \rangle \qquad A^* = A.$$

$$= \langle \lambda \vec{x}, \vec{x} \rangle$$

$$= \lambda^* \langle \vec{x}, \vec{x} \rangle$$

$$= \lambda^* \|\vec{x}\|^2.$$

Summary:

$A^* = A$ & $A\vec{x} = \lambda \vec{x} \Longrightarrow \lambda \|\vec{x}\|^2 = \lambda^* \|\vec{x}\|^2$

If $\vec{x} \neq \vec{0}$ then $\|\vec{x}\|^2 \neq 0$

hence $\lambda = \lambda^*$.

i.e., λ is a <u>real</u> number.

$B = A^* A$ (pos. semidef)

Since $B^* = (A^* A)^*$

$$= A^* A^{**}$$

$= A^* A = B,$

we know eigenvalues λ are <u>real</u>.

Moreover, I claim that eigenvalues are non-neg: $\lambda \geq 0$.

<u>Proof</u>: Assume $B\vec{x} = \lambda \vec{x}$, $\vec{x} \neq \vec{0}$.

$$\begin{aligned}
\lambda \|\vec{x}\|^2 &= \lambda \langle \vec{x}, \vec{x} \rangle \\
&= \langle \vec{x}, \lambda \vec{x} \rangle \\
&= \langle \vec{x}, B\vec{x} \rangle \\
&= \langle \vec{x}, A^* A \vec{x} \rangle \\
&= \langle A\vec{x}, A\vec{x} \rangle \\
&= \|A\vec{x}\|^2.
\end{aligned}$$

$$\lambda \|\vec{x}\|^2 = \|A\vec{x}\|^2$$

$$\vec{x} \neq \vec{0} \implies \|\vec{x}\|^2 \neq 0$$

$$\lambda = \|A\vec{x}\|^2 / \|\vec{x}\|^2 \geq 0.$$

One more step.

Note: A^{-1} does not exist

$\iff$ $\det(A) = 0$

$\iff$ $\det(0I - A) = 0$

$\iff$ 0 is an e.value of A.

$B = A^*A$, A has ind cols,

then B^{-1} exists hence

e.values of $B > 0$.

We can turn this around.

Theorem: Let $B^* = B$. Then

$B = A^*A$ for some A

$\iff$ e.values of B are real & ≥ 0.

Can detect definiteness from the eigenvalues.

Proof: This is hard. It will follow from the spectral theorem.

[Preview:

$B^* = B$ with ≥ 0 evalues.

S.T. $\Rightarrow$ $B = U \Lambda U^*$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Makes sense

$$\text{"}\sqrt{\Lambda}\text{"} = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}.$$

real non-neg.

Take $A^* = U\sqrt{\Lambda}$

so $A^*A = U\sqrt{\Lambda}\,(\cancel{\sqrt{\Lambda}})^* U^*$

$\sqrt{\Lambda}$

$$= U \Lambda U^* = B. \quad]$$

Problem: Show

$B^* = B \implies B$ has an orthonormal basis of e.vectors.

<u>The Spectral Theorem.</u>