Higher order diff eq in 1 variable, constant coeffs:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_{n-1} y(t) = 0.$$

Linear Algebra:

$$\vec{x}(t) = \left(y(t), y'(t), \dots, y^{(n-1)}(t) \right)^T.$$

$(n-1) \times (n-1)$

$$\vec{x}\,'(t) = \underbrace{\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -a_{n-1} & \dots & \dots & -a_2 & -a_1 \end{pmatrix}}_{\text{Companion matrix } C} \vec{x}(t)$$

$$\vec{x}\,'(t) = C \vec{x}(t)$$

Unique solution: $\vec{x}(t) = \exp(Ct)\,\vec{x}(0)$

Problem: Compute $\exp(Ct)$ for companion matrices C.

There are <u>explicit</u> formulas for the Jordan form of C.

Key:

$$\det(xI - C)$$

$$= x^n + a_1 x^{n-1} + \cdots + a_{n-1}$$

$$= \prod (x - \lambda_i)^{n_i}$$

C has Jordan blocks $J_{n_i}(\lambda_i)$.

Small example:

$$y''(t) + a y'(t) + b y(t) = 0.$$

no constant term.

$$\vec{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}, \quad \vec{x}'(t) = \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix}$$

$$\vec{x}'(t) = \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

$$\chi\begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} = \det \begin{pmatrix} x & -1 \\ b & x+a \end{pmatrix}$$

$$= x(x+a) + b$$

$$= x^2 + ax + b$$

$\lambda^2 + a\lambda + b = 0$

$-b - \lambda a = \lambda^2$

$$= (x-\lambda)(x-\mu).$$

TWO CASES:

$\lambda \neq \mu$: Check $\begin{pmatrix} 1 \\ \lambda \end{pmatrix}, \begin{pmatrix} 1 \\ \mu \end{pmatrix}$ are the eigenvectors of C.

$$\begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ -b - \lambda a \end{pmatrix}$$

$$= \begin{pmatrix} \lambda \\ \lambda^2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \lambda \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda & \mu \end{pmatrix} \underbrace{\begin{pmatrix} \lambda & \\ & \mu \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda & \mu \end{pmatrix}^{-1}}_{\text{just involves the eigenvalues!}}$$

$$\exp(Ct) = \begin{pmatrix} 1 & 1 \\ \lambda & \mu \end{pmatrix} \begin{pmatrix} e^{\lambda t} & \\ & e^{\mu t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda & \mu \end{pmatrix}^{-1}$$

Get clean solution by changing coordinates:

$$\vec{u}(t) = \begin{pmatrix} 1 & 1 \\ \lambda & \mu \end{pmatrix}^{-1} \vec{x}(t)$$

solution

$$\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \vec{x}(t) = \exp(Ct)\,\vec{x}(0)$$

$$= \begin{pmatrix} 1 & 1 \\ \lambda & \mu \end{pmatrix} \begin{pmatrix} e^{\lambda t} & \\ & e^{\mu t} \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}$$

$\rightsquigarrow$

$$\boxed{y(t) = u_1(0)\, e^{\lambda t} + u_2(0)\, e^{\mu t}.}$$

Case $\lambda = \mu$:

Then $C = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$ is <u>not</u> diagonalizable. Jordan form.

$$C = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}^{-1}$$

Jordan block.

$$\exp(Ct) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}^{-1}$$

Change coords:

$$\vec{u}(t) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}^{-1} \vec{x}(t).$$

Solution:

$$\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \vec{x}(t) = \exp(Ct)\,\vec{x}(0)$$

$$= \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}.$$

$$\rightsquigarrow$$

$$\boxed{y(t) = u_1(0)\, e^{\lambda t} + u_2(0)\, \underline{t}\, e^{\lambda t}}$$

In the notes:

$$\exp\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t \right)$$

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$$\exp\left(\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} t \right)$$

$$\exp\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} bt\right) = \begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}.$$

Markov Matrix:
(stochastic)

$$M = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}$$

has entries $0 \le m_{ij} \le 1$,
and each column sums to 1.

$$(1 \cdots 1)\, M = 1 \cdot (1 \cdots 1)$$

$$M^T \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

So 1 is an e.value of M^T.

But M, M^T have the same characteristic polynomial, hence same eigenvalues. So $\exists\ \vec{x} \neq \vec{0}$

$$M\vec{x} = 1\vec{x}.$$

But we don't know what $\vec{x}$ is.

A few cases:

- Absorbing Markov Chain.

Can find a permutation matrix X such that

$$M = X \left(\begin{array}{c|c} I & R \\ \hline O & Q \end{array} \right) X^{-1}$$

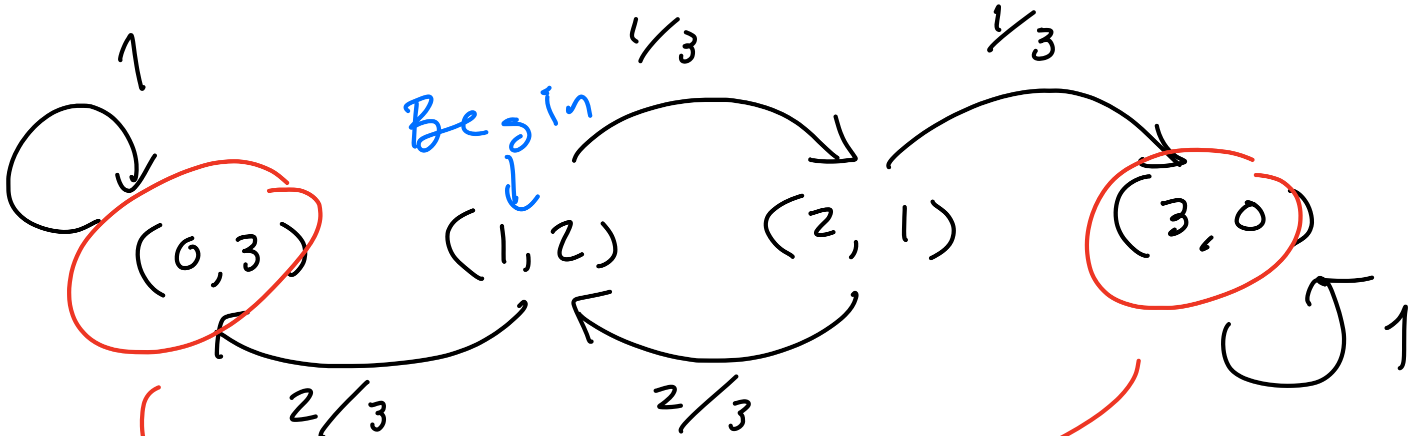
where entries of Q are < 1.

Then from HW 2:

$$M^n = X \left(\begin{array}{c|c} I & R(I+Q+\cdots+Q^{n-1}) \\ \hline 0 & Q^n \end{array} \right) X^{-1}$$

$$\to X \left(\begin{array}{c|c} I & R(I-Q)^{-1} \\ \hline 0 & 0 \end{array} \right) X^{-1}$$

e.g. Gambler's Ruin (HW 6).



absorbing states.
you can never escape.

Question: If you begin in

state $(1,2)$,

$P(\text{end in state } (0,3)) = ?$

$P(\text{end in state } (3,0)) = ?$

One of the motivating problems in history of probability.

Solutions are the entries of the matrix $R(I-Q)^{-1}$.

Harder Case:

Some power of M has all entries > 0.

Corollary of "Perron-Frobenius":

There exists a unique vector $\vec{p}$ satisfying

- $M\vec{p} = 1\vec{p}$
- entries of $\vec{p}$ sum to 1

$$(1\ 1\ \cdots\ 1)\,\vec{p} = 1.$$

- All entries of $\vec{p}$ are > 0.
- Every other eigenvalue of M has $|\lambda| < 1$. (need not be real).

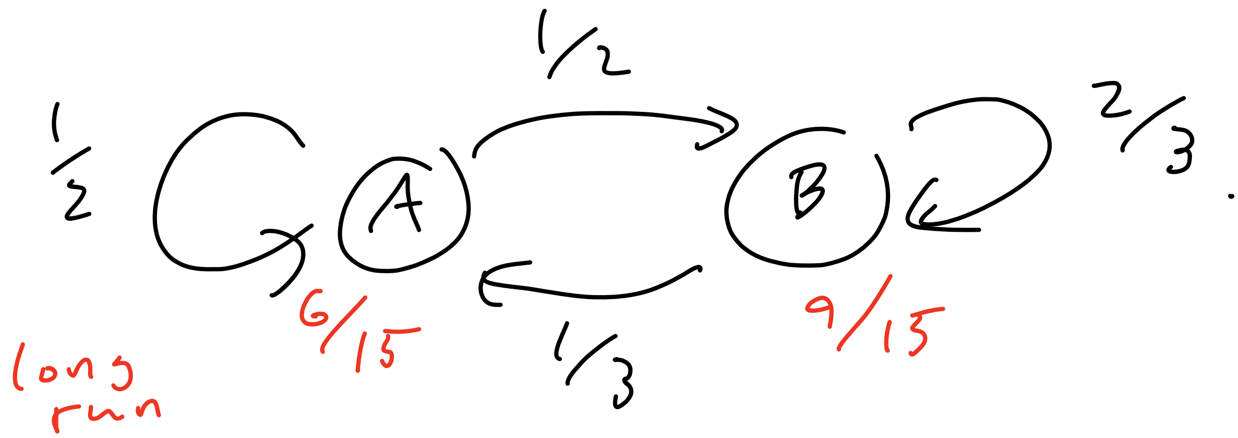
Applying this (HW6) you can show that

$$M^n \longrightarrow (\vec{p}\,|\,\vec{p}\,|\cdots|\,\vec{p})$$

$$\vec{p}\,(1\ \cdots\ 1)$$

as $n \to \infty$.

e.g. $M = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix}$.



$$\begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = 1 \begin{pmatrix} p \\ q \end{pmatrix}.$$

$$p + q = 1.$$

Solution:

$$\begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix} \begin{pmatrix} 2/3 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$$

$$\vec{p} = \frac{3}{5} \begin{pmatrix} 2/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6/15 \\ 9/15 \end{pmatrix}.$$