Why bother with the abstract definition of vector spaces?

Example: $L^2[0,1]$ = the set of real-valued functions $f: [0,1] \to \mathbb{R}$ satisfying

$$\int_0^1 f(x)^2 \, dx \quad \text{exists.}$$

Define addition & scalar mult of functions: Given $a \in \mathbb{R}$, $f(x), g(x)$

$$(f+g)(x) = f(x) + g(x)$$
$$(af)(x) = a f(x).$$

Define an inner product:

$$\langle f(x), g(x) \rangle = \int_0^1 f(x) g(x) \, dx$$

Claim: These 3 operations make $L^2[0,1]$ into an inner-product space over $\mathbb{R}$.

Recall:

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

$$\langle \vec{u}, a\vec{v} \rangle = a \langle \vec{u}, \vec{v} \rangle = \langle a\vec{u}, \vec{v} \rangle$$

$$\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle.$$

$$\langle \vec{v}, \vec{v} \rangle \geq 0$$

$$\langle \vec{v}, \vec{v} \rangle = 0 \iff \vec{v} = \vec{0}.$$

Check:

$$\langle f(x), f(x) \rangle = \int_0^1 f(x) f(x)\, dx$$

$$= \int_0^1 f(x)^2\, dx \geq 0$$

$$\int_0^1 f(x)^2\, dx = 0$$

$$\implies f(x) = 0$$

Another reason for abstraction.

$\mathbb{R}^n$ has a "standard basis"

$$\vec{e}_1 = (1, 0, \ldots, 0)$$
$$\vec{e}_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$
$$\vec{e}_n = (0, \ldots, 0, 1)$$

Property: Every $\vec{v} \in \mathbb{R}^n$ has the form

$$\vec{v} = \underbrace{a_1 \vec{e}_1 + a_2 \vec{e}_2 + \cdots + a_n \vec{e}_n}_{\text{called "linear combination"}}$$

for some unique scalars $a_1, \ldots, a_n$.

Proof is silly: Let $\vec{v} = (v_1, \ldots, v_n)$

Then

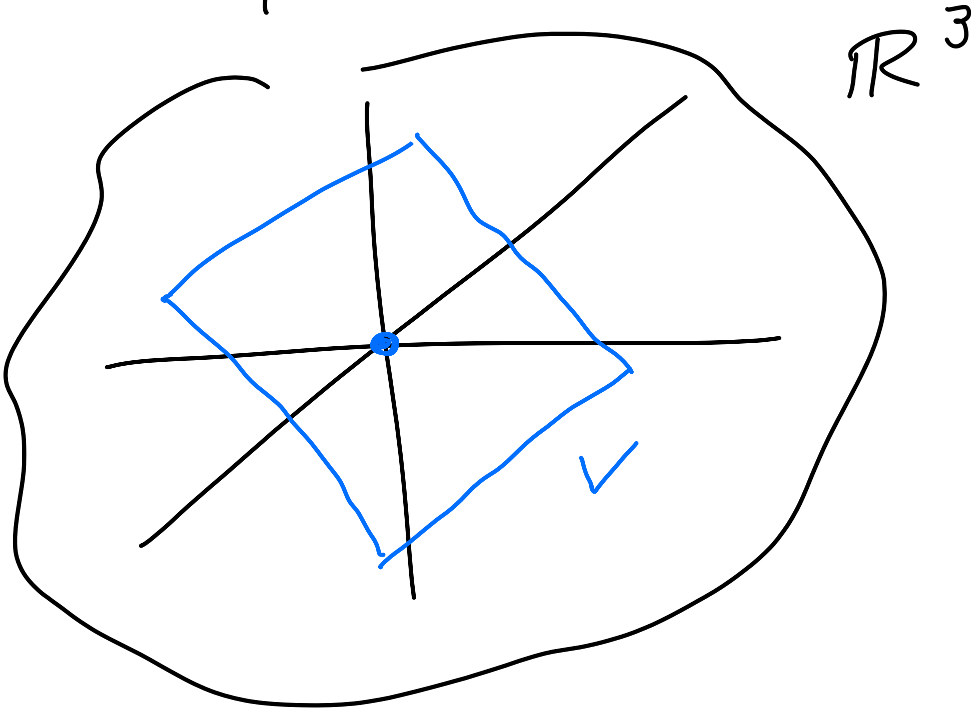
$$\begin{aligned}\vec{v} &= (v_1, 0, \dots, 0) \\ &\quad + (0, v_2, 0, \dots, 0) \\ &\quad \vdots \\ &\quad + (0, 0, \dots, 0, v_n) \\ &= v_1 (1, 0, \dots, 0) \\ &\quad + v_2 (0, 1, 0, \dots, 0) \\ &\quad \vdots \\ &\quad + v_n (0, \dots, 0, 1) \\ &= v_1 \vec{e_1} + v_2 \vec{e_2} + \dots + v_n \vec{e_n}\end{aligned}$$

Next, say $V \subseteq \mathbb{R}^n$ is a <u>subspace</u> if

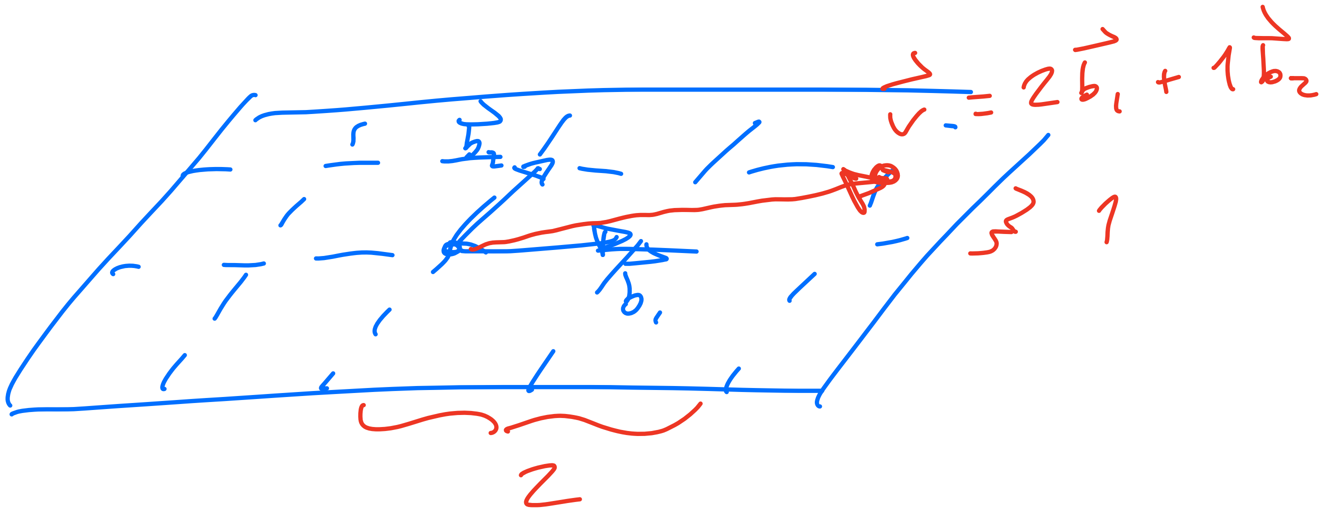
- $\vec{0} \in V$
- $a \in \mathbb{R}, \vec{v} \in V \implies a\vec{v} \in V$
- $\vec{u}, \vec{v} \in V \implies \vec{u} + \vec{v} \in V$.

e.g. A <u>line</u> through the origin is a subspace.

Also, a plane through the origin is a subspace.



Problem: Subspace V does not have a "standard basis". So we must pick an arbitrary basis. What we want:



Want two vectors $\vec{b}_1, \vec{b}_2 \in V$ so that for <u>any</u> $\vec{v} \in V$ we can write

$$\vec{v} = a_1 \vec{b}_1 + a_2 \vec{b}_2$$

for some <u>unique</u> scalars $a_1, a_2 \in \mathbb{R}$.

Another e.g.

Let $\vec{b}_1 = (1,1,1)$

$\vec{b}_2 = (1,2,3)$

$$\vec{v} = 3(1,1,1) - 2(1,2,3)$$

$$= (1,-1,-3).$$

$\vec{v}$ has standard coords $(1,-1,-3)$ in space $\mathbb{R}^3$.

But $\vec{v}$ has coordinates $(3,-2)$ as an element of the plane V with respect to the basis $\vec{b}_1, \vec{b}_2$.

Given general vector space V, what is a "basis"?

Consider a set of vectors

$$B = \{ \vec{b}_1, \vec{b}_2, \dots, \vec{b}_m \} \subseteq V.$$

- Say B is a spanning set if for all $\vec{v} \in V$, there exist at least one choice of scalars $a_1, \dots, a_m$ such that

$$\vec{v} = a_1 \vec{b}_1 + a_2 \vec{b}_2 + \dots + a_m \vec{b}_m \qquad (*)$$

- Say B is (linearly) independent if for all $\vec{v} \in V$, there is <u>at most</u> <u>one</u> choice of scalars $a_1, \dots, a_m$ satisfying $(*)$.

A basis is a set $B \subseteq V$ that is spanning & independent.

i.e. every $\vec{v} \in V$ has exactly one expression of the form

$$\vec{v} = a_1 \vec{b}_1 + a_2 \vec{b}_2 + \cdots + a_m \vec{b}_m$$

In this case we say $(a_1, \ldots, a_m)$ the "B-coordinates" of $\vec{v}$.

$$\vec{v} = (a_1, a_2, \ldots, a_m)_B$$

in B-coordinates.

Important Theorem:

If vector space V has a basis of size n, then any basis of V has size n. In which case we say

$$\boxed{\dim V = n.}$$

Proof : HW 1.

Example : Standard basis of $\mathbb{R}^n$ has n elements $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ hence $\dim \mathbb{R}^n = n$.

Question :

$$\dim L^2[0,1] = ?$$

Preview : $\dim L^2[0,1] = \infty$.

But it still has "bases"

e.g. the set

$$1, \ \sin(2\pi k x), \ \cos(2\pi k x)$$

for $k = 1, 2, 3, \dots$

is a "basis" for $L^2[0,1]$.