MTH 510: Linear Algebra II

Goal: Cover the interesting applications

- Perron-Frobenius Theorem.
  - Google's Page Rank algorithm
  - Markov chains
  - Applications to random walks
- Least-Squares Approximation

$$A\vec{x} = \vec{b} \rightsquigarrow A^T A \vec{x} = A^T \vec{b}$$

Applies to regression in probability & statistics.

- Singular Value Decomposition.
  - image compression
  - data compression.
  - also called PCA (principal component analysis)

- Fast Fourier Transform.

  - used in signal transmission (cell phones, etc.)

  - gives the fastest way to multiply numbers inside a computer

  - Fourier transform in general is one of the most useful techniques in physics.

Begin with a quick review of MTH 210.

The prototype: Euclidean Space.

Let $\mathbb{R}$ be the set of real numbers.
Let $\mathbb{R}^n$ be the set of ordered n-tuples of real numbers:

$$\mathbb{R}^n = \{ (x_1, \ldots, x_n) : x_i \in \mathbb{R} \}$$

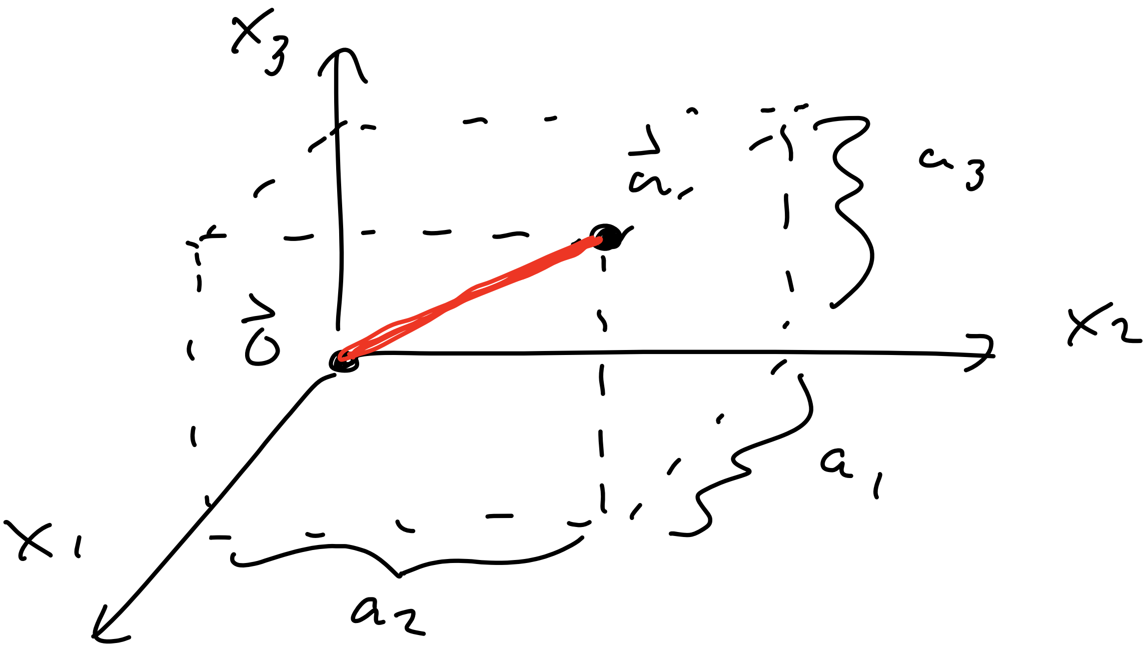
We can think of $\mathbb{R}^n$ as the set of <u>points</u> in "n-dimensional space". We usually write

$$\vec{x} = (x_1, x_2, \ldots, x_n).$$

There is a special point called the "origin" $\vec{0} = (0, 0, \ldots, 0)$.

Descartes 1637

Space $\longleftrightarrow$ Numbers

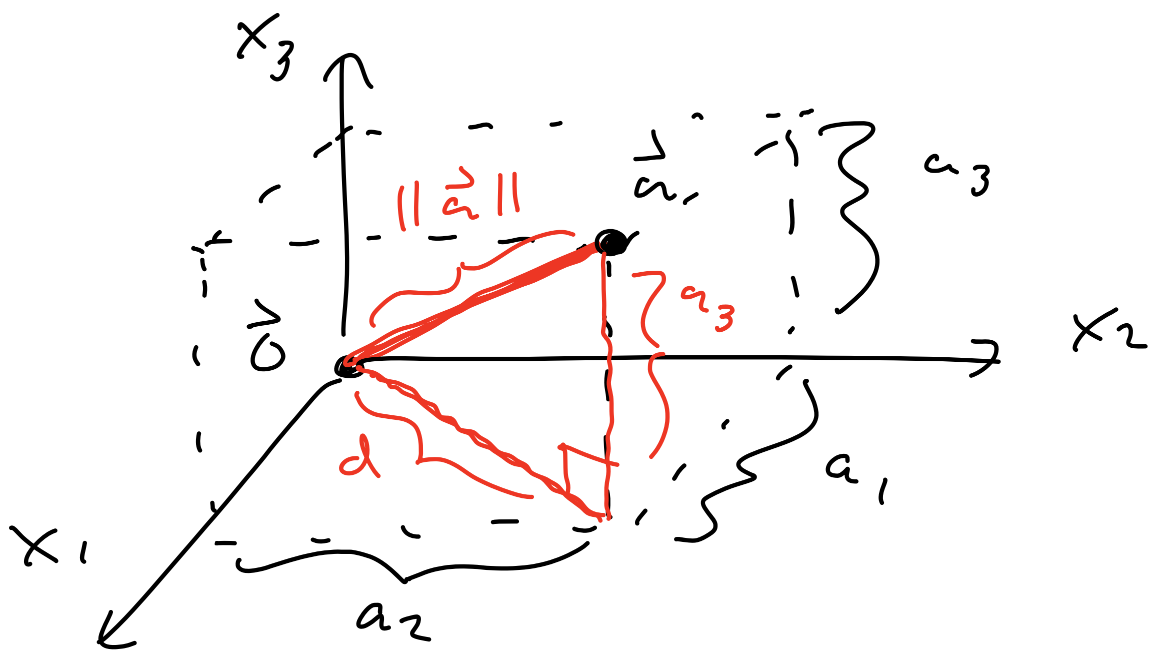


Point $\vec{\ } = (a_1, a_2, a_3)$ is at the opposite corner of the rectangular box of dimensions $a_1 \times a_2 \times a_3$ from the origin $\vec{O}$.

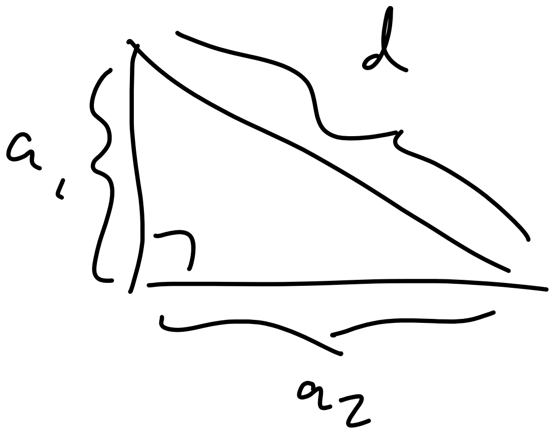
Can use the Pythagorean Theorem to compute distance from $\vec{O}$:

$$\|\vec{a}\| := \mathrm{dist}(\vec{O}, \vec{a})$$
$$= \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Proof for $n = 3$.



We have two right triangles:

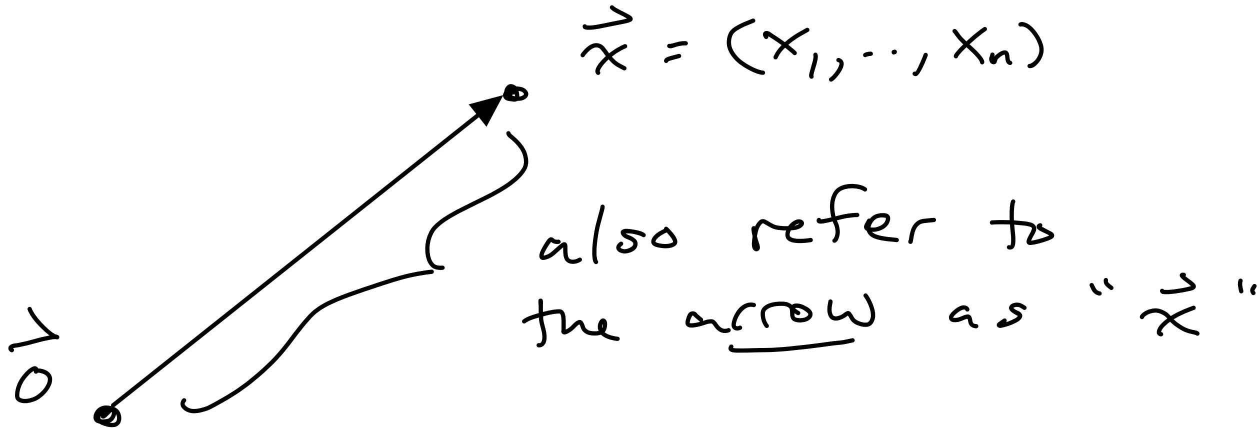


$$d^2 = a_1^2 + a_2^2$$

$$\|\vec{a}\|^2 = d^2 + a_3^2$$

$$\|\vec{a}\|^2 = d^2 + a_3^2$$
$$= (a_1^2 + a_2^2) + a_3^2$$
$$= a_1^2 + a_2^2 + a_3^2 \quad \checkmark$$

We can also view elements of the set $\mathbb{R}^n$ as directed line segments ("arrows"):



In Calc textbooks they write

$$\vec{x} = \langle x_1, \dots, x_n \rangle$$

when referring to the arrow, but advanced textbooks don't bother.

Theorem (Parallelogram Law):

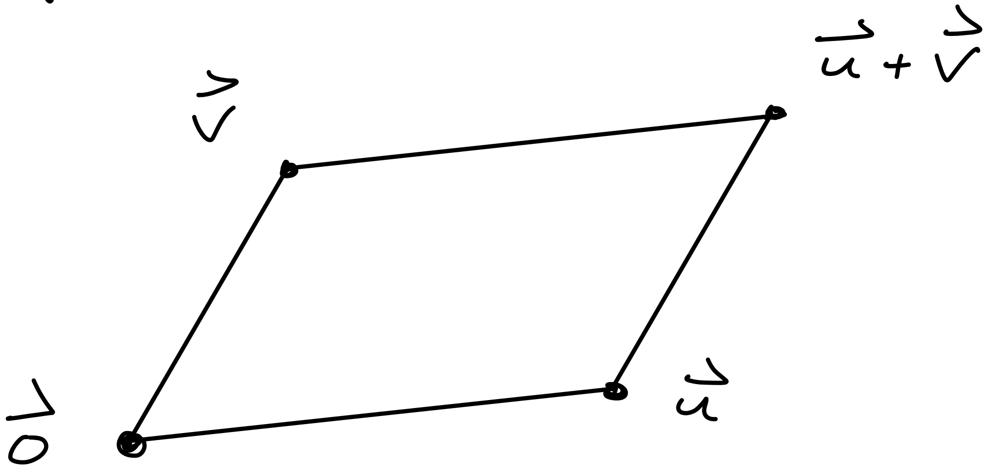
Given points $\vec{u} = (u_1, \dots, u_n)$
$\vec{v} = (v_1, \dots, v_n)$

we define a new point by adding the coordinates:

$$\text{"}\vec{u} + \vec{v}\text{"} := (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n).$$

Geometrically, the four points

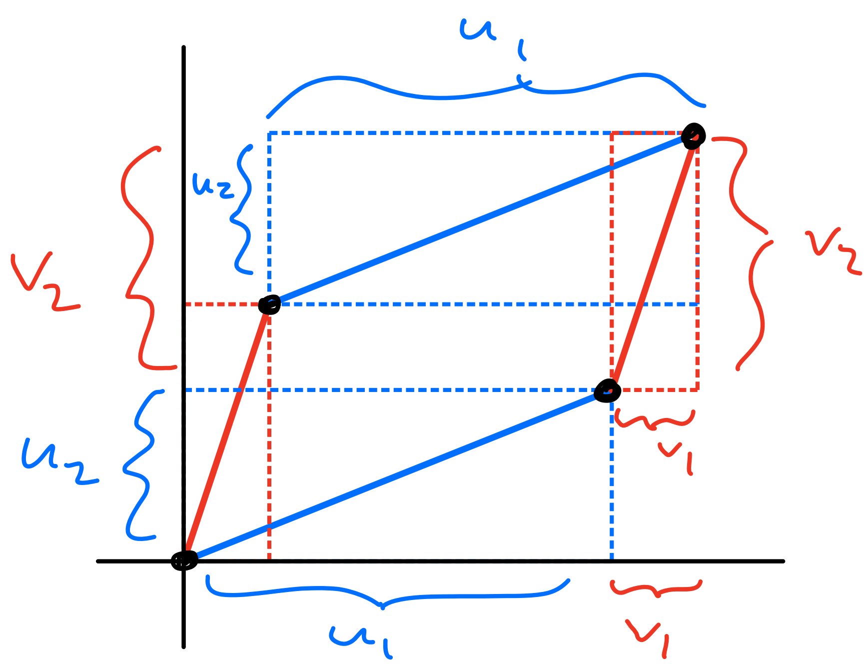
$\vec{0}, \vec{u}, \vec{v}, \vec{u}+\vec{v}$ are the vertices of a parallelogram:



Geometry $\longleftrightarrow$ Algebra

parallelogram $\quad$ addition

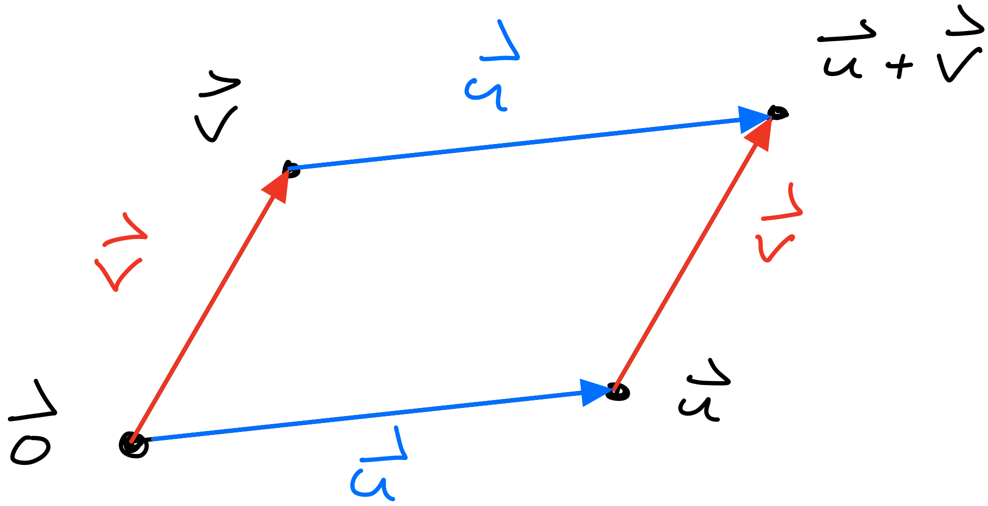
Proof for $n = 2$:



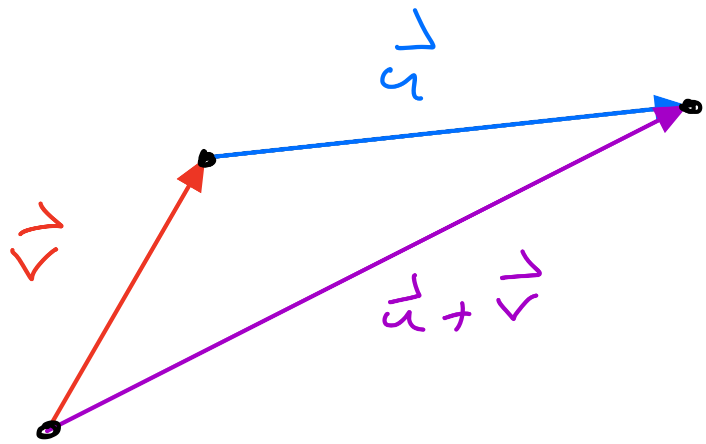
Subtle Issue:

points vs. arrows.

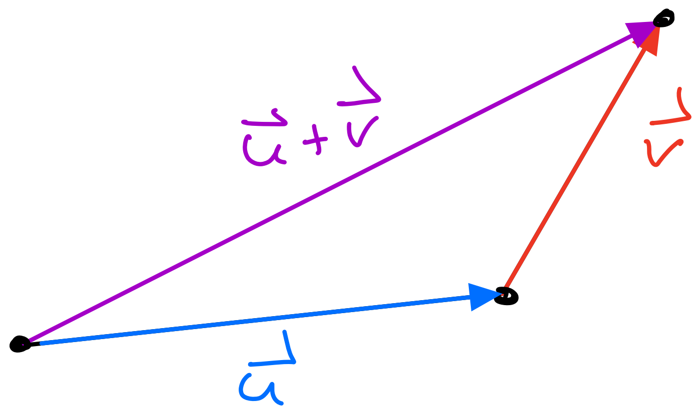
We regard two arrows as <u>equal</u> when they have the same "direction & magnitude", i.e., they form opposite sides of a parallelogram:



With this in mind, we can "add arrows" as follows:



arrows add head-to-tail



note that $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

Note: We defined points, then arrows, so

arrow = ordered pair of points

arrow = "head minus tail"

In this language the arrow $\vec{0}$ has head = tail.

i.e., $\vec{0}$ is the unique arrow with "length zero".

We'll discuss more next time.