Math 510 **Fall 2022**
Homework 3 **Drew Armstrong**

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1. The Important Matrices A^TA and AA^T. Let A be any $m \times n$ matrix. Consider the $n \times n$ (symmetric) matrix A^TA and the $m \times m$ (symmetric) matrix AA^T.

(a) Show that $N(A^TA) = N(A)$. [Hint: Use the trick formula $\mathbf{x}^TA^TA\mathbf{x} = \|A\mathbf{x}\|^2$.]

(b) Use part (a) to show that

$$\text{rank}(A^TA) = \text{rank}(A) = \text{rank}(AA^T).$$

[Hint: The Fundamental Theorem says that $\text{rank}(A) = \text{rank}(A^T)$.]

(c) If A has independent columns, prove that $(A^TA)^{-1}$ exists. If A has independent rows, prove that $(AA^T)^{-1}$ exists. [Hint: Use part (b).]

2. Projection Matrices. A square matrix P is called a *projection matrix* when

$$P^2 = P \quad \text{and} \quad P^T = P.$$

(a) If P is a projection matrix, show that $P + (I - P) = I$ and $P(I - P) = O$.

(b) If P is a projection matrix, show that $I - P$ is also a projection matrix.

(c) Let P be a projection matrix. For any vector $\mathbf{x}$ we define $\mathbf{x}_1 = P\mathbf{x}$ and $\mathbf{x}_2 = (I - P)\mathbf{x}$. Show that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ and $\mathbf{x}_1^T\mathbf{x}_2 = 0$. This is the geometric meaning of projection.

(d) Let A be any matrix with independent columns, so that $(A^TA)^{-1}$ exists. Check that the following matrix is a projection matrix:

$$P = A(A^TA)^{-1}A^T.$$

[Remark: This matrix projects onto the column space of A.]

(e) Use part (d) to find the 3×3 matrix that projects onto the plane $x - 2y + 1z = 0$. [Hint: It is easier to find the matrix P that projects onto the orthogonal complement, which is the line spanned by $(1, -2, 1)$. Then the projection onto the plane is $I - P$.]

3. CMR Factorization. Let A be any $m \times n$ matrix of rank r. Pick any r columns of A that form a basis for the column space and call them $\mathbf{c}_1, \ldots, \mathbf{c}_r \in \mathbb{R}^m$. Pick any r rows of A that form a basis for the row space and call them $\mathbf{r}_1, \ldots, \mathbf{r}_r \in \mathbb{R}^n$. Define the matrices

$$C = \begin{pmatrix} | & & | \\ \mathbf{c}_1 & \cdots & \mathbf{c}_r \\ | & & | \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} - & \mathbf{r}_1^T & - \\ & \vdots & \\ - & \mathbf{r}_r^T & - \end{pmatrix}.$$

(a) Prove that there exists an invertible $r \times r$ matrix M such that $A = CMR$. [Hint: We know from Problem 1 that $(C^TC)^{-1}$ and $(RR^T)^{-1}$ exist. Check that $M = (CC^T)^{-1}(C^TAR^T)(RR^T)^{-1}$ works.]

(b) Compute a CMR factorization for the rank 1 matrix

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}.$$

(c) Compute a CMR factorization for the rank 2 matrix

$$A = \begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix}.$$

[Hint: Use the first two columns and the first two rows.]

4. Distance Between Skew Lines. Consider two lines $(1,0,0)+s(1,2,1)$ and $(1,1,1)+t(1,1,1)$ living in $\mathbb{R}^3$.

(a) Suppose that the lines intersect, so that $(1,0,0)+s(1,2,1)=(1,1,1)+t(1,1,1)$ for some values of s and t. Express this as a single matrix equation:

$$A\begin{pmatrix}s\\t\end{pmatrix}=\mathbf{b},$$

(b) If the lines **don't** intersect then the matrix equation in part (a) has no solution. In this case, find the least squares solution for (s,t).

5. Bilinear Forms. Given a matrix $B\in\mathbb{R}^n$ we define a function $\langle -,-\rangle_B:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ by

$$\langle \mathbf{x},\mathbf{y}\rangle_B := \mathbf{x}^T B\mathbf{y}.$$

Remark: If I is the identity matrix then $\langle \mathbf{x},\mathbf{y}\rangle_I$ is just the dot product on $\mathbb{R}^n$.

(a) Show that this function is bilinear.

(b) Let $\mathbf{e}_1,\ldots,\mathbf{e}_n\in\mathbb{R}^n$ be the standard basis. In this case show that

$$\langle \mathbf{e}_i,\mathbf{e}_j\rangle_B = (ij \text{ entry of } B).$$

(c) For any two $n\times n$ matrices B and C, show that

$$B=C \quad\iff\quad \langle \mathbf{x},\mathbf{y}\rangle_B=\langle \mathbf{x},\mathbf{y}\rangle_C \text{ for all } \mathbf{x},\mathbf{y}\in\mathbb{R}^n.$$

[Hint: One direction uses part (b).]

(d) **Symmetric.** Show that $B=B^T$ if and only if $\langle \mathbf{x},\mathbf{y}\rangle_B=\langle \mathbf{y},\mathbf{x}\rangle_B$ for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$.

(e) **Semi-Definite.** If $B=A^TA$ for some rectangular $A\in\mathbb{R}^{m\times n}$, show that

$$\langle \mathbf{x},\mathbf{x}\rangle_B\geq 0 \text{ for all } \mathbf{x}\in\mathbb{R}^n.$$

(f) **Definite.** If $B=A^TA$ for some A with **independent columns**, show that

$$\langle \mathbf{x},\mathbf{x}\rangle_B=0 \quad\iff\quad \mathbf{x}=\mathbf{0}.$$

[Hint for parts (e) and (f): Use the trick formula $\langle \mathbf{x},\mathbf{x}\rangle_B=\|A\mathbf{x}\|^2$.]

6. Orthogonal Subspaces (Optional). Let V be a Hermitian inner product space. For any subspace $U\subseteq V$ we define its *orthogonal complement*:

$$U^\perp := \{\mathbf{v}\in V:\langle \mathbf{u},\mathbf{v}\rangle=0 \text{ for all } \mathbf{u}\in U\}.$$

(a) Prove that $U^\perp$ is also a subspace of V.

(b) Prove that $U\cap U^\perp=\{\mathbf{0}\}$.

(c) If U is finite dimensional with basis $\{\mathbf{u}_1,\ldots,\mathbf{u}_m\}$, show that

$$\mathbf{v}\in U^\perp \quad\iff\quad \langle \mathbf{u}_i,\mathbf{v}\rangle=0 \text{ for all } i.$$

(d) If U is finite dimensional, prove that $U+U^\perp=V$, which means that any vector $\mathbf{v}\in V$ can be expressed as $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$ for some $\mathbf{v}_1\in U$ and $\mathbf{v}_2\in U^\perp$. [Hint: Let $\{\mathbf{u}_1,\ldots,\mathbf{u}_m\}$ be an orthonormal basis for U and define the *projection function* $\pi:V\to U$ by

$$\pi(\mathbf{v})=\sum_{i=1}^m \langle \mathbf{u}_i,\mathbf{v}\rangle\mathbf{u}_i.$$

For any $\mathbf{v}\in V$, use part (c) to show that $\mathbf{v}-\pi(\mathbf{v})\in U^\perp$.]

(e) Combine (b) and (d) to prove that $U\oplus U^\perp=V$, which means that any vector $\mathbf{v}\in V$ can be expressed as $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$ for some **unique** $\mathbf{v}_1\in U$ and $\mathbf{v}_2\in U^\perp$.

(f) If V is finite dimensional, prove that $\dim U + \dim U^{\perp} = \dim V$. [Hint: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be an orthonormal basis for U and let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be an orthonormal basis for $U^{\perp}$. Use part (e) to prove that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a basis for V.]