**Math 510/610** **Exam 2**
**Fall 2022** **Fri Nov 4**

No electronic devices are allowed. There are 4 pages and each page is worth 6 points, for a total of 24 points.

**1. Linear and Bilinear Forms.** Consider the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^2$.

(a) Let $\varphi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the bilinear form defined by

$$\varphi(\mathbf{e}_1, \mathbf{e}_1) = 0, \quad \varphi(\mathbf{e}_1, \mathbf{e}_2) = 1, \quad \varphi(\mathbf{e}_2, \mathbf{e}_1) = -1, \quad \varphi(\mathbf{e}_2, \mathbf{e}_2) = 0.$$

Use this information to compute $\varphi(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} = (1, 2)$ and $\mathbf{y} = (2, -1)$.

We have

$$\begin{aligned} \varphi(\mathbf{x}, \mathbf{y}) &= \varphi(1\mathbf{e}_1 + 2\mathbf{e}_2, 2\mathbf{e}_1 - 1\mathbf{e}_2) \\ &= 2 \cdot \varphi(\mathbf{e}_1, \mathbf{e}_1) - 1 \cdot \varphi(\mathbf{e}_1, \mathbf{e}_2) + 4 \cdot \varphi(\mathbf{e}_2, \mathbf{e}_1) - 2 \cdot \varphi(\mathbf{e}_2, \mathbf{e}_2) \\ &= 2 \cdot 0 - 1 \cdot 1 + 4 \cdot (-1) - 2 \cdot 0 \\ &= -5. \end{aligned}$$

Alternatively, the provided information tells us that

$$\begin{aligned} \varphi(\mathbf{x}, \mathbf{y}) &= \mathbf{x}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{y} \\ &= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \end{pmatrix} \\ &= -1 - 4 \\ &= -5. \end{aligned}$$

Remark: This bilinear form is secretly the determinant of a $2 \times 2$ matrix:

$$\begin{aligned} \varphi \left( \begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \end{array} \right) &= \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ &= a_1 b_2 - a_2 b_1. \end{aligned}$$

(b) Express the following polynomial in the form $f(\mathbf{x}) = b + \mathbf{b}^T\mathbf{x} + \mathbf{x}^T B \mathbf{x}$ for some scalar $b \in \mathbb{R}$, vector $\mathbf{b} \in \mathbb{R}^2$ and **symmetric** matrix $B \in \mathbb{R}^{2\times 2}$:

$$f(\mathbf{x}) = f(x, y) = 2 + x + 5y + x^2 + xy - 3y^2.$$

*Solution.*

$$f(x, y) = 2 + \begin{pmatrix} 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & 1/2 \\ 1/2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

**2. Fundamental Subspaces.** Consider the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

(a) Compute the dimensions of the four fundamental subspaces: $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\mathcal{C}(A)$, $\mathcal{N}(A^T)$. [You do not need to find bases for these subspaces; just the dimensions.]

*Solution.* The spaces $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are orthogonal complements in $\mathbb{R}^3$, so that

$$\dim \mathcal{R}(A) + \dim \mathcal{N}(A) = 3.$$

The spaces $\mathcal{C}(A)$ and $\mathcal{N}(A^T)$ are orthogonal complements in $\mathbb{R}^2$, so that

$$\dim \mathcal{C}(A) + \dim \mathcal{N}(A^T) = 2.$$

Since the two rows of $A$ are not parallel we have $\dim \mathcal{R}(A) = 2$ and hence $\dim \mathcal{N}(A) = 1$. Finally, from the Fundamental Theorem we have $\dim \mathcal{C}(A) = \dim \mathcal{R}(A) = 2$ and hence $\dim \mathcal{N}(A^T) = 0$.

(b) Since $A$ has independent rows and is not square, we know that it has infinitely many right inverses. Find one. [Hint: There is a shortcut using $(AA^T)^{-1}$.]

*Shortcut.* Since $A$ has independent rows we know that $(AA^T)^{-1}$ exists. But then $B = A^T(AA^T)^{-1}$ is a right inverse of $A$ because

$$AB = A[A^T(AA^T)^{-1}] = (AA^T)(AA^T)^{-1} = I.$$

Computation:

$$\begin{aligned} B &= A^T(AA^T)^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix} \\ &= \frac{1}{6} \begin{pmatrix} 5 & -3 \\ 2 & 0 \\ -1 & 3 \end{pmatrix}. \end{aligned}$$

*The Long Way.* Let $X = \left( \mathbf{x} \mid \mathbf{y} \right)$ be a $3 \times 2$ matrix satisfying $AX = I_2$, which is equivalent to the matrix equations $A\mathbf{x} = \mathbf{e}_1$ and $A\mathbf{y} = \mathbf{e}_2$. We can solve both systems simultaneously by row-reducing an augmented matrix:

$$\left( \begin{array}{ccc|c|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{array} \right) \rightsquigarrow \left( \begin{array}{ccc|c|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 2 & 0 & 1 \end{array} \right).$$

Thus we have

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

which gives

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Hence the general right inverse of $A$ has the form

$$X = \begin{pmatrix} 1+s & -1+t \\ -2s & 1-2t \\ s & t \end{pmatrix}.$$

The shortcut answer corresponds to $s = -1/6$ and $t = 1/2$.

**3. Least Squares.** Consider the following three data points in the $x, y$-plane:

$$(x_1, y_1) = (0, 1), \quad (x_2, y_2) = (1, 3), \quad (x_3, y_3) = (2, 3).$$

We wish to find the line $y = a + bx$ thst is closest to these data points.

(a) Write a single matrix equation $X\mathbf{a} = \mathbf{y}$ for the unknowns $\mathbf{a} = (a, b)$ to express the fact that all three data points are on the line $y = a + bx$.

We have

$$\left\{ \begin{array}{ll} a + x_1 b & = y_1 \\ a + x_2 b & = y_2 \\ a + x_3 b & = y_3 \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{ll} a + 0b & = 1 \\ a + 1b & = 3 \\ a + 2b & = 3 \end{array} \right\} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}.$$

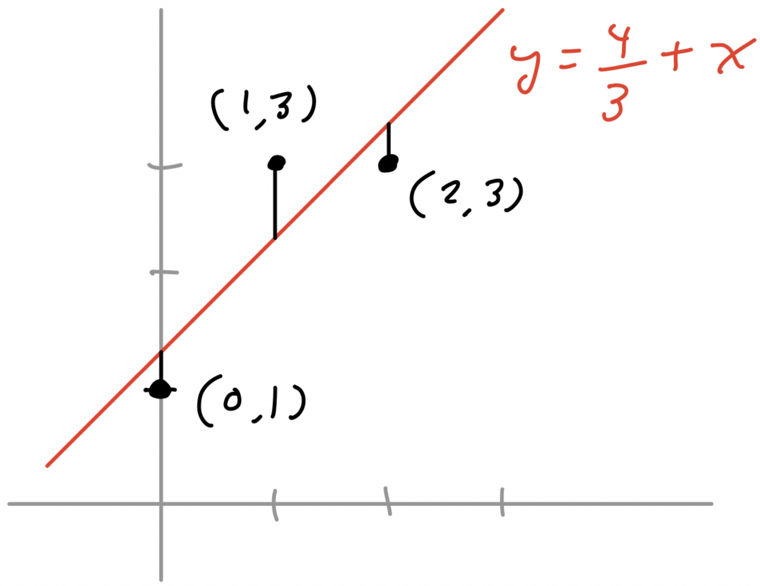
Remark: This is the same matrix from Problem 2.

(b) The equation from part (a) has no solution. Instead, solve the normal equation $X^T X\mathbf{a} = X^T\mathbf{y}$ to find the best fit line.

We have

$$\begin{aligned} X\mathbf{a} &= \mathbf{y} \\ X^T X\mathbf{a} &= X^T\mathbf{y} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 7 \\ 9 \end{pmatrix} \\ \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 9 \end{pmatrix} \\ &= \frac{1}{6} \begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ 9 \end{pmatrix} \\ &= \frac{1}{6} \begin{pmatrix} 8 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}. \end{aligned}$$

So the best fit line is $y = 4/3 + 1x$. Here is a picture:




**4. Determinants.** Let $A$ and $B$ be square matrices of the same size. You can assume the following properties of determinants:

- $\det(A) \neq 0$ if and only if $A^{-1}$ exists,
- $\det(I) = 1$,
- $\det(A^T) = \det(A)$,
- $\det(AB) = \det(A)\det(B)$.

Use these to prove the following results.

(a) If $A^T A = I$ then $\det(A) = \pm 1$.

*Proof.* Suppose that $A^T A = I$. Then we have

$$\begin{aligned} A^T A &= I \\ \det(A^T A) &= \det(I) \\ \det(A^T)\det(A) &= 1 \\ \det(A)\det(A) &= 1 \\ \det(A)^2 &= 1 \\ \det(A) &= \pm 1. \end{aligned}$$

(b) If $(AB)^{-1}$ exists, then $A^{-1}$ and $B^{-1}$ exist.

*Proof.* We have

$$\begin{aligned} (AB)^{-1} \text{ exists} &\implies \det(AB) \neq 0 \\ &\implies \det(A)\det(B) \neq 0 \\ &\implies \det(A) \neq 0 \text{ and } \det(B) \neq 0 \\ &\implies A^{-1} \text{ exists and } B^{-1} \text{ exists.} \end{aligned}$$

(c) If $A^{-1}$ exists then $\det(A^{-1}BA) = \det(B)$.

*Proof.* Suppose that $A^{-1}$ exists, so that $\det(A) \neq 0$. Then since $A^{-1}A = I$ we have

$$\begin{aligned} \det(A^{-1}A) &= \det(I) \\ \det(A^{-1})\det(A) &= 1 \\ \det(A^{-1}) &= 1/\det(A), \end{aligned}$$

and for any matrix $B$ we have

$$\begin{aligned} \det(A^{-1}BA) &= \det(A^{-1})\det(B)\det(A) \\ &= \frac{1}{\det(A)}\det(B)\det(A) \\ &= \frac{1}{\det(A)}\det(A)\det(B) \\ &= \det(B). \end{aligned}$$

(Matrix multiplication is not commutative, but determinants are scalars.)