Math 510/610 **Exam 1**
Fall 2022 **Fri Sept 30**

No electronic devices are allowed. There are 4 pages and each page is worth 6 points, for a total of 24 points.

1. Inner Products. Let V be a real inner product space with norm $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

(a) Suppose that vectors $\mathbf{u}, \mathbf{v} \in V$ satisfy $\langle \mathbf{u}, \mathbf{u} \rangle = 2$, $\langle \mathbf{v}, \mathbf{v} \rangle = 3$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 2$. Use this information to compute the distance $\|\mathbf{u} - \mathbf{v}\|$.

We have

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &= \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle - 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= 2 - 2 \cdot 2 + 3 \\ &= 1, \end{aligned}$$

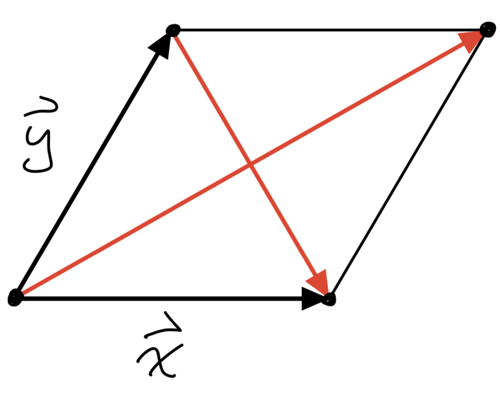
and hence $\|\mathbf{u} - \mathbf{v}\| = 1$.

(b) Suppose that vectors $\mathbf{x}, \mathbf{y} \in V$ satisfy $\|\mathbf{x}\| = \|\mathbf{y}\|$. In this case prove that vectors $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are orthogonal, i.e., that their inner product is zero.

Assuming that $\|\mathbf{x}\| = \|\mathbf{y}\|$, the inner product of $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ is

$$\begin{aligned} \langle \mathbf{x} + \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle &= \langle \mathbf{x}, \mathbf{x} \rangle - \cancel{\langle \mathbf{x}, \mathbf{y} \rangle} + \cancel{\langle \mathbf{x}, \mathbf{y} \rangle} - \langle \mathbf{y}, \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{y}, \mathbf{y} \rangle \\ &= \cancel{\|\mathbf{x}\|^2} - \cancel{\|\mathbf{y}\|^2} \\ &= 0. \end{aligned}$$

Example: In Euclidean space, the diagonals of a rhombus are perpendicular.



2. Linear Functions. Let $\bullet$ denote the dot product on $\mathbb{R}^3$. Given the column vector $\mathbf{a} = (1, 2, 3) \in \mathbb{R}^3$, we define the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(\mathbf{x}) = (\mathbf{a} \bullet \mathbf{x})\mathbf{a}$.

(a) Prove that the function f is linear.

For all vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^3$ and scalars $c_1, \ldots, c_n \in \mathbb{R}$, the properties of dot product and scalar multiplication give

$$\begin{aligned} f(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n) &= (\mathbf{a} \bullet (c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n))\mathbf{a} \\ &= (c_1\mathbf{a} \bullet \mathbf{x}_1 + \cdots + c_n\mathbf{a} \bullet \mathbf{x}_n)\mathbf{a} \\ &= c_1(\mathbf{a} \bullet \mathbf{x}_1)\mathbf{a} + \cdots + c_n(\mathbf{a} \bullet \mathbf{x}_n)\mathbf{a} \\ &= c_1 f(\mathbf{x}_1) + \cdots + c_n f(\mathbf{x}_n). \end{aligned}$$

(b) Find the associated 3×3 matrix $[f]$.

It follows from part (a) that the function f can be represented as a 3×3 matrix. There are two ways to find this matrix.

Direct Way. Recall that the jth column of the matrix $[f]$ is defined to be $f(\mathbf{e}_j)$. In our case we have

$$f(\mathbf{e}_1) = \left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

and

$$f(\mathbf{e}_1) = \left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

and

$$f(\mathbf{e}_3) = \left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}.$$

Hence

$$[f] = \begin{pmatrix} | & | & | \\ f(\mathbf{e}_1) & f(\mathbf{e}_2) & f(\mathbf{e}_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}.$$

Clever Way. For any vector $\mathbf{x} \in \mathbb{R}^3$ we have

$$\begin{aligned} f(\mathbf{x}) &= (\mathbf{a} \bullet \mathbf{x})\mathbf{a} & \\ &= (\mathbf{a}^T\mathbf{x})\mathbf{a} & \text{dot product as row times column} \\ &= \mathbf{a}(\mathbf{a}^T\mathbf{x}) & \text{scalars commute with vectors} \\ &= (\mathbf{a}\mathbf{a}^T)\mathbf{x}. & \text{associativity of matrix product} \end{aligned}$$

It follows that[1]

$$[f] = \mathbf{a}\mathbf{a}^T = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}.$$

[1] We have shown that $[f]\mathbf{x} = (\mathbf{a}\mathbf{a}^T)\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^3$. In particular, this implies that the jth columns are the same: $[f]\mathbf{e}_j = (\mathbf{a}\mathbf{a}^T)\mathbf{x}$. Hence the matrices are the same $[f] = \mathbf{a}\mathbf{a}^T$.

3. Matrix Multiplication. Let A be an $m \times n$ matrix with column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ and let B be any $n \times m$ matrix. Express each answer as a matrix product.

The goal is to think of these computations at the matrix level, not the scalar level.

(a) Give an expression for the ij entry of $A^T A$.

$$\begin{aligned}(ij \text{ entry of } A^T A) &= (i\text{th row of } A^T)(j\text{th column of } A)\\ &= (i\text{th column of } A)^T(j\text{th column of } A)\\ &= \mathbf{a}_i^T \mathbf{a}_j.\end{aligned}$$

(b) Give an expression for the jth column of BA.

$$(j\text{th column of } B) = B(j\text{th column of } A) = B\mathbf{a}_j.$$

(c) Give an expression for the ij entry of $(BA)^T(BA)$. [Hint: Combine (a) and (b).]

$$\begin{aligned}(ij \text{ entry of } (BA)^T(BA)) &= (i\text{th row of } (BA)^T)(j\text{th column of } BA)\\ &= (i\text{th column of } BA)^T(j\text{th column of } BA)\\ &= (B\mathbf{a}_i)^T(B\mathbf{a}_j)\\ &= \mathbf{a}_i^T B^T B\mathbf{a}_j.\end{aligned}$$

4. Symmetric Matrices. We say that a (square) matrix S is symmetric when $S^T = S$. We say that S is antisymmetric when $S^T = -S$.

(a) For any matrix A (possibly non-square), prove that $S = A^T A$ is symmetric.

$$S^T = (A^T A)^T = A^T (A^T)^T = A^T A = S.$$

(b) Prove that any **square** matrix A is a sum of a symmetric and an antisymmetric matrix. [Hint: Consider the matrices $A + A^T$ and $A - A^T$.]

We observe that $A + A^T$ is symmetric:

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T.$$

While $A - A^T$ is antisymmetric:

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T).$$

Scaling these matrices by $1/2$ does not change the symmetry or antisymmetry, hence we can express A as a sum of a symmetric and an antisymmetric matrix:

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T).$$