MTH 505: Number Theory **Spring 2020**
Homework 4 **Drew Armstrong**

Problem 1. Computing Legendre Symbols. Use Quadratic Reciprocity and its supplements to compute the Legendre symbol (47/67). [Hint: 47 and 67 are prime.]

Problem 2. Quadratic Character of -2. Let p be an odd prime. We proved in class that

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p = 1 \bmod 4, \\ -1 & \text{if } p = 3 \bmod 4, \end{cases} \quad \text{and} \quad \left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p = 1, 7 \bmod 8, \\ -1 & \text{if } p = 3, 5 \bmod 8. \end{cases}$$

Compute the Legendre symbol $(-2/p)$. [Hint: We know that $(-2/p) = (-1/p)(2/p)$.]

Problem 3. Quadratic Character of 3. For any odd prime p, Quadratic Reciprocity says

$$\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right)(-1)^{\frac{p-1}{2}\frac{3-1}{2}} = \left(\frac{p}{3}\right)(-1)^{\frac{p-1}{2}}.$$

Use this to compute the Legendre symbol $(3/p)$. [Hint: First observe that $(p/3) = 1$ when $p = 1 \bmod 3$ and $(p/3) = -1$ when $p = 2 \bmod 3$. Observe also that $(-1)^{(p-1)/2} = 1$ when $p = 1 \bmod 4$ and $(-1)^{(p-1)/2}$ when $p = 3 \bmod 4$. Now use the Chinese Remainder Theorem.]

Problem 4. Infinitely Many Primes $= 3$ mod 8. Let $p_1, \ldots, p_k$ be a set of primes such that $p_i = 3 \bmod 8$ for all i, and consider the number

$$N = (p_1 \cdots p_k)^2 + 2.$$

We will use this to show that there exists a prime number $p = 3 \bmod 8$ that is not in the list.

(a) Show that $N = 2 \bmod 8$.
(b) Show that every prime divisor $p|N$ satisfies $p = 1$ or $p = 3 \bmod 8$. [Hint: If $p|N$ then show that $-2 = (p_1 \cdots p_k)^2 \bmod p$. Now use Problem 2.]
(c) Combine (a) and (b) to show that there exists a prime divisor $p|N$ satisfying $p = 3 \bmod 8$. [Hint: If all prime divisors $= 1 \bmod 8$ then $N = 1 \bmod 8$.]
(d) Show that the prime p from part (c) is not in the list $p_1, \ldots, p_k$. [Hint: $N = 2 \bmod p_i$.]