Math 461 Homework 4 — Spring 2023 Drew Armstrong

1. Conjugation of Complex Polynomials. For any polynomial $f(x) = \sum \alpha_k x^k \in \mathbb{C}[x]$ we define the *conjugate polynomial* by taking the complex conjugate of each coefficient:

$$f^*(x) := \sum \alpha_k^* x^k \in \mathbb{C}[x].$$

(a) For any complex polynomial $f(x) \in \mathbb{C}[x]$ and complex number $\alpha \in \mathbb{C}$, show that

$$[f(\alpha)]^* = f^*(\alpha^*).$$

(b) For any complex polynomial $f(x) \in \mathbb{C}[x]$ show that

$$f(x) = f^*(x) \quad \Longleftrightarrow \quad f(x) \in \mathbb{R}[x].$$

(c) For any complex polynomials $f(x), g(x) \in \mathbb{C}[x]$, show that

$$(f+g)^*(x) = f^*(x) + g^*(x) \quad \text{and} \quad (fg)^*(x) = f^*(x)g^*(x).$$

(d) For any complex polynomial $f(x) \in \mathbb{C}[x]$, show that

$$f(x) + f^*(x) \in \mathbb{R}[x] \quad \text{and} \quad f(x)f^*(x) \in \mathbb{R}[x].$$

(a): Consider any $f(x) = \sum \beta_k x^k \in \mathbb{C}[x]$ and $\alpha \in \mathbb{C}$. Then we have

$$\begin{aligned}
[f(\alpha)]^* &= \left[\sum \beta_k (\alpha)^k\right]^* \\
&= \sum \beta_k^* (\alpha^*)^k && \text{properties of } * \\
&= f^*(\alpha^*).
\end{aligned}$$

(b): Consider any $f(x) = \sum \alpha_k x^k$. Then we have

$$\begin{aligned}
f(x) \in \mathbb{R}[x] \quad &\Longleftrightarrow \quad \alpha_k \in \mathbb{R} \text{ for all } k \\
&\Longleftrightarrow \quad \alpha_k = \alpha_k^* \in \mathbb{R} \text{ for all } k \\
&\Longleftrightarrow \quad \sum \alpha_k x^k = \sum \alpha_k^* x^k \\
&\Longleftrightarrow \quad f(x) = f^*(x).
\end{aligned}$$

(c): Consider any $f(x) = \sum \alpha_k x^k \in \mathbb{C}[x]$ and $g(x) = \sum \beta_k x^k \in \mathbb{C}[x]$. Then we have

$$\begin{aligned}
(f+g)^*(x) &= \left[\sum (\alpha_k + \beta_k) x^k\right]^* \\
&= \sum (\alpha_k + \beta_k)^* x^k && \text{definition} \\
&= \sum (\alpha_k^* + \beta_k^*) x^k && \text{property of } * \\
&= \sum \alpha_k^* x^k + \sum \beta_k^* x^k \\
&= f^*(x) + g^*(x)
\end{aligned}$$

and

$$(fg)^*(x) = \left[\sum \left(\sum_{i+j=k} \alpha_i \beta_j\right) x^k\right]^*$$

$$\begin{aligned}
&= \sum \left(\sum_{i+j=k} \alpha_i \beta_j \right)^* x^k && \text{definition} \\
&= \sum \left(\sum_{i+j=k} \alpha_i^* \beta_j^* \right) x^k && \text{properties of } * \\
&= \left(\sum \alpha_i^* x^i \right) \left(\sum \beta_j^* x^j \right) \\
&= f^*(x) g^*(x).
\end{aligned}$$

(d): This can be done directly by looking at the coefficients, but it is quicker to combine (b) and (c). Given $f(x) \in \mathbb{C}[x]$, part (c) tells us that

$$(f + f^*)^* = f^* + f^{**} = f^* + f = f + f^*$$

and

$$(ff^*)^* = f^* f^{**} = f^* f = ff^*.$$

(The notation is the tricky part here.) Then part (b) tells us that $f + f^* \in \mathbb{R}[x]$ and $ff^* \in \mathbb{R}[x]$.

Remark: Let $f(x) = \sum \alpha_k x^k \in \mathbb{C}[x]$. The ugly proof of $f(x)f^*(x) \in \mathbb{R}[x]$ shows that every coefficient of $f(x)f^*(x)$ is equal to its own conjugate:

$$\left(\sum_{i+j=k} \alpha_i \alpha_j^* \right)^* = \sum_{i+j=k} \alpha_i^* \alpha_j^{**} = \sum_{i+j=k} \alpha_i^* \alpha_j = \sum_{i+j=k} \alpha_j \alpha_i^* = \sum_{i+j=k} \alpha_i \alpha_j^*.$$

The last equality switches the indices i and j and uses the fact that $i + j = j + i$.

2. Invariance of Quotient and Remainder. Consider two fields $\mathbb{F} \subseteq \mathbb{E}$, so we can think of $\mathbb{F}[x]$ is a subring of $\mathbb{E}[x]$.

(a) Given $f(x), g(x) \in \mathbb{F}[x]$ such that $g(x) \neq 0$, suppose we have polynomials $q(x), r(x)$ with coefficients in $\mathbb{E}$ satisfying

$$\begin{cases} f(x) = q(x)g(x) + r(x), \\ r(x) = 0 \text{ or } \deg(r) < \deg(g). \end{cases}$$

In this case, prove that $q(x)$ and $r(x)$ actually have coefficients in $\mathbb{F}$. [Hint: Divide $f(x)$ by $g(x)$ in the ring $\mathbb{F}[x]$ to obtain a quotient and remainder $q'(x), r'(x) \in \mathbb{F}[x]$. Now use the **uniqueness** of quotient and remainder in the ring $\mathbb{E}[x]$ (HW1.3).]

(b) Now consider the field extension $\mathbb{C} \supseteq \mathbb{R}$. Suppose that the real polynomial $f(x) \in \mathbb{R}[x]$ has a complex root $\alpha \in \mathbb{C}$ that is not real (i.e., $\alpha \neq \alpha^*$). In this case, prove that there exists a **real** polynomial $h(x) \in \mathbb{R}[x]$ satisfying

$$f(x) = (x - \alpha)(x - \alpha^*)h(x).$$

(a): Recall the result of Problem 3 on Homework 1: *Let $\mathbb{E}$ be a field and consider polynomials $f(x), g(x) \in \mathbb{E}[x]$ with $g(x) \neq 0$.* **If** *there exist $q(x), q'(x), r(x), r'(x) \in \mathbb{E}[x]$ satisfying*

$$\begin{cases} f(x) = g(x)q(x) + r(x), \\ r(x) = 0 \text{ or } \deg(r) < \deg(g), \end{cases} \qquad \begin{cases} f(x) = g(x)q'(x) + r'(x), \\ r'(x) = 0 \text{ or } \deg(r') < \deg(g), \end{cases}$$

then *we must have $r(x) = r'(x)$ and $q(x) = q'(x)$.*

To apply this to the current problem, consider two fields $\mathbb{E} \supseteq \mathbb{F}$ and two polynomials $f(x), g(x) \in \mathbb{F}[x]$ with $g(x) \neq 0$. Suppose we are given polyonomials $q(x), r(x) \in \mathbb{E}[x]$ satisfying

$$\begin{cases} f(x) = g(x)q(x) + r(x), \\ r(x) = 0 \text{ or } \deg(r) < \deg(g), \end{cases}$$

In this case, our goal is to show that $q(x), r(x) \in \mathbb{F}[x]$. Since $f(x), g(x)$ have coefficients in $\mathbb{F}$ we can apply long division in the ring $\mathbb{F}[x]$ to obtain $q'(x), r'(x) \in \mathbb{F}[x]$ satisfying

$$\begin{cases} f(x) = g(x)q'(x) + r'(x), \\ r'(x) = 0 \text{ or } \deg(r') < \deg(g). \end{cases}$$

Since $\mathbb{F}[x] \subseteq \mathbb{E}[x]$, all of the six polynomials f, g, q, q', r, r' are in $\mathbb{E}[x]$. Hence it follows from the above result that $r(x) = r'(x)$ and $q(x) = q'(x)$. Finally, since q', r' have coefficients in $\mathbb{F}$, it follows that q, r have coefficients in $\mathbb{F}$.

Remark: This result is a bit subtle. I taught this material a few times before I appreciated it.

(b): Suppose that a real polynomial $f(x) \in \mathbb{R}[x]$ has a complex root $\alpha \in \mathbb{C}$. By Descartes' Theorem there exists a polynomial $g(x) \in \mathbb{C}[x]$ with complex coefficients such that

$$f(x) = (x - \alpha)g(x).$$

Since $f(x)$ has real coefficients, we also have

$$0 = 0^* = [f(\alpha)]^* = f^*(\alpha^*) = f(\alpha^*).$$

And since α is not real we have $\alpha \neq \alpha^*$. Putting these these facts together gives

$$\begin{aligned} (x - \alpha)g(x) &= f(x) \\ (\alpha^* - \alpha)g(\alpha^*) &= f(\alpha^*) \\ (\alpha^* - \alpha)g(\alpha^*) &= 0 \\ g(\alpha^*) &= 0. \end{aligned}$$

Applying Descartes' Theorem again, there exists a polynomial $h(x) \in \mathbb{C}[x]$ with complex coefficients such that

$$g(x) = (x - \alpha^*)h(x).$$

Now we have

$$\begin{aligned} f(x) &= (x - \alpha)g(x) \\ &= (x - \alpha)(x - \alpha^*)h(x) \\ &= (x - (\alpha + \alpha^*)x + \alpha\alpha^*)h(x) \\ &= p(x)h(x), \end{aligned}$$

where $p(x)$ has real coefficients. Finally, we will apply part (a) to show that $h(x)$ has real coefficients. Indeed, since $f(x), p(x) \in \mathbb{R}[x]$ there exist $q(x), r(x) \in \mathbb{R}[x]$ satisfying

$$\begin{cases} f(x) = p(x)q(x) + r(x), \\ r(x) = 0 \text{ or } \deg(r) < \deg(p). \end{cases}$$

On the other hand, we have

$$\begin{cases} f(x) = p(x)h(x) + 0, \\ 0 = 0 \text{ or } \deg(0) < \deg(p). \end{cases}$$

It follows that $r(x) = 0$ and $q(x) = h(x)$. In particular, $h(x)$ has real coefficients.

Remark: Maybe there is an easier way to prove this? I don't know. Anyway, the result is basic and important. We will use it in 3(a).

3. Equivalent Forms of the FTA.

(a) Suppose that every (non-constant) real polynomial $f(x) \in \mathbb{R}[x]$ has a root in $\mathbb{C}$. Use Problem 2 to prove that every (non-constant) real polynomial factors as a product of real polynomials of degrees 1 and 2.

(b) Suppose that every (non-constant) real polynomial $f(x) \in \mathbb{R}[x]$ has a root in $\mathbb{C}$. In this case, use Problem 1 to prove that every (non-constant) complex polynomial $f(x) \in \mathbb{C}[x]$ has a root in $\mathbb{C}$. [Hint: Let $f(x) \in \mathbb{C}[x]$ be any (non-constant) complex polynomial and consider the polynomial $g(x) = f(x)f^*(x)$.]

(a): Suppose that every non-constant polynomial $f(x) \in \mathbb{R}[x]$ has a complex root. Our goal is to prove that every non-constant polynomial $f(x) \in \mathbb{R}[x]$ has the desired type of factorization.

So consider any non-constant polynomial $f(x) \in \mathbb{R}[x]$. By assumption, there exists $\alpha \in \mathbb{C}$ such that $f(\alpha) = 0$. There are two cases:

- If α is real then there exists a real polynomial $g(x) \in \mathbb{R}[x]$ such that

$$f(x) = (x - \alpha)g(x).$$

Note that $x - \alpha$ is real of degree 1. If $g(x)$ is constant then we are done. Otherwise, we may assume for induction that $g(x)$ has the desired type of factorization. Then so does $f(x)$.

- If α is not real then from Problem 2(b) we have

$$f(x) = p(x)h(x)$$

for some real polynomials $p(x), h(x)$ with $\deg(p) = 2$. If $h(x)$ is constant then we are done. Otherwise, we may assume for induction that $h(x)$ has the desired type of factorization. Then so does $f(x)$.

$\square$

Remark: This is the grown-up way of doing induction, i.e., we don't say that we're using induction until the very end. If you insist on being explicit, we are using strong induction to prove that the following statement is true for all $n \geq 1$:

> $P_n :=$ "Every real polynomial of degree n can be factored as a product of real polynomials of degrees 1 and 2".

The base cases $n = 1, 2$ are easy. Now assume that $P_1, P_2, \ldots, P_{n-1}$ are true. In order to prove P_n, consider any $f(x) \in \mathbb{R}[x]$ of degree $n \geq 2$. The constructed polynomials $g(x)$ and $h(x)$ have degrees $n-1$ and $n-2$. Since P_{n-1} and P_{n-2} are true, each of them has the desired type of factorization.

(b): Suppose that every non-constant polynomial $f(x) \in \mathbb{R}[x]$ has a complex root. We wish to prove that every non-constant polynomial $f(x) \in \mathbb{C}[x]$ has a complex root.

So consider any non-constant polynomial $f(x) \in \mathbb{C}[x]$ and define the polynomial

$$g(x) = f(x)f^*(x),$$

which is also non-constant. In fact $\deg(g) = 2\deg(f)$. From 1(d) we know that $g(x)$ has real coefficients, hence by assumption $g(x)$ has a complex root, say $\alpha \in \mathbb{C}$. But then

$$\begin{aligned} f(x)f^*(x) &= g(x) \\ f(\alpha)f^*(\alpha) &= g(\alpha) \\ f(\alpha)f^*(\alpha) &= 0, \end{aligned}$$

which implies that $f(\alpha) = 0$ or $f^*(\alpha) = 0$. If $f(\alpha) = 0$ then $f(x)$ has a complex root α and we are done. Otherwise, if $f^*(\alpha) = 0$ then

$$f(\alpha^*) = f^{**}(\alpha^*) = [f^*(\alpha^*)]^* = 0^* = 0.$$

Hence $f(x)$ has a complex root α^*. $\square$

4. Biquadratic Polynomials. Given real numbers $a, b \in \mathbb{R}$ we will prove that there exist real numbers $p, q, r, s \in \mathbb{R}$ satisfying

$$x^4 + ax^2 + b = (x^2 + px + r)(x^2 + qx + s).$$

(a) If $a^2 - 4b \geq 0$, show that $x^4 + ax^2 + b = (x^2 - A)(x^2 - B)$ for some real $A, B \in \mathbb{R}$.
(b) If $a^2 - 4b < 0$, show that $x^4 + ax^2 + b = (x^2 - \alpha)(x^2 - \alpha^*)$ for some non-real $\alpha \in \mathbb{C}$.
(c) Continuing from (b), let $\pm\beta \in \mathbb{C}$ be the roots of $x^2 - \alpha \in \mathbb{C}[x]$ and let $\pm\gamma \in \mathbb{C}$ be the roots of $x^2 - \alpha^* \in \mathbb{C}[x]$, so that

$$x^4 + ax^2 + b = (x - \beta)(x + \beta)(x - \gamma)(x + \gamma).$$

Show that $\beta^* = \gamma$ or $\beta^* = -\gamma$. Now what?

(a): Given any real numbers $a, b \in \mathbb{R}$ we consider the polynomials

$$f(x) = x^4 + ax^2 + b \quad \text{and} \quad g(x) = x^2 + ax + b.$$

By the quadratic formula, we can write $g(x) = (x - A)(x - B)$, where

$$A, B = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

If $a^2 - 4b \geq 0$ then both of these roots are real (possibly equal), and we conclude that

$$f(x) = g(x^2) = (x^2 - A)(x^2 - B)$$

for some real numbers A, B.

(b): If $a^2 - 4b < 0$ then I claim that $A, B = \alpha, \alpha^*$ for some non-real complex number α. Indeed, in this case we have

$$A, B = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{-a}{2} \pm i\frac{1}{2}\sqrt{4b - a^2},$$

where $\sqrt{4b - a^2}$ is real because $4b - a^2 > 0$.

(c): Continuing from (b), we have shown that

$$f(x) = g(x^2) = (x^2 - \alpha)(x^2 - \alpha^*)$$

for some non-real complex number α. Recall that any nonzero complex number α has two square roots $\pm\beta \in \mathbb{C}$ which are negatives of each other. Indeed, a square root $\beta^2 = \alpha$ must exist.[1] Then we also have $(-\beta)^2 = \beta^2 = \alpha$. And we cannot have more than two square roots

[1] We can show this using the polar form. If $\alpha = re^{i\theta}$ with $r, \theta \in \mathbb{R}$ and $r > 0$ then we can take $\beta = \sqrt{r} \cdot e^{i\theta/2}$, where $\sqrt{r}$ is the positive real square root of r. If you want to be really thorough, the square root $\sqrt{r}$ exists because of the Intermediate Value Theorem.

because Descartes' Theorem says that the polynomial $x^2 - \alpha$ has at most two roots. Similarly, the number α^* has exactly two square roots $\pm\gamma \in \mathbb{C}$. It follows from Descartes' Theorem that

$$f(x) = (x^2 - \alpha)(x^2 - \alpha^*) = (x - \beta)(x + \beta)(x - \gamma)(x + \gamma).$$

Finally, we will show that $\beta^* = \gamma$ or $\beta^* = -\gamma$. Indeed, since $\beta^2 = \alpha$ we have

$$\begin{aligned} \beta^2 &= \alpha \\ (\beta^2)^* &= \alpha^* \\ (\beta^*)^2 &= \alpha^*. \end{aligned}$$

Since $\pm\gamma$ are the only square roots of α^* it follows that β^* equals one of these. **In either case**, we have

$$\begin{aligned} f(x) &= (x - \beta)(x + \beta)(x - \gamma)(x + \gamma) \\ &= (x - \beta)(x - \beta^*)(x + \beta)(x + \beta^*) \\ &= (x^2 - (\beta + \beta^*)x + \beta\beta^*)(x^2 + (\beta + \beta^*)x + \beta\beta^*), \end{aligned}$$

where all of the coefficients are real. □

Remark: This proof was abstract. Actually, the coefficients can be computed explicitly. In the course notes there is a section on "Euler's Attempt", which give the following example:

$$x^4 - 4x^2 + 7 = \left(x^2 + \sqrt{4 + 2\sqrt{7}} \cdot x + \sqrt{7}\right)\left(x^2 - \sqrt{4 + 2\sqrt{7}} \cdot x + \sqrt{7}\right).$$

This is a special feature of biquadratic polynomials $x^4 + ax^2 + b$. The factorization of a general quartic $x^4 + ax^3 + bx^2 + cx + d$ cannot be easily described.

5. Laplace's Proof of the FTA (Bonus). Consider any field extension $\mathbb{E} \supseteq \mathbb{R}$ and let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{E}$ be any elements satisfying

$$x^3 + x + 1 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3).$$

(a) Expand the right hand side and compare coefficients to find formulas for

$$\alpha_1 + \alpha_2 + \alpha_3, \quad \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, \quad \text{and} \quad \alpha_1\alpha_2\alpha_3.$$

(b) Now let $t \in \mathbb{R}$ be any real number and consider the polynomial

$$g_t(x) = (x - \beta_{12t})(x - \beta_{13t})(x - \beta_{23t}),$$

where $\beta_{ijt} = \alpha_i + \alpha_j + t\alpha_i\alpha_j$ for all pairs $1 \leq i < j \leq 3$. Find explicit formulas for the coefficients of $g_t(x)$ in terms of t, and conclude that $g_t(x)$ has **real coefficients**. [Hint: Each coefficient g_t is a symmetric function of $\alpha_1, \alpha_2, \alpha_3$. Any symmetric function of $\alpha_1, \alpha_2, \alpha_3$ can be expressed in terms of the *elementary symmetric functions* of $\alpha_1, \alpha_2, \alpha_3$, as listed in part (a).]

(a): Expanding the right hand side gives

$$\begin{aligned} x^3 + x + 1 &= (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \\ &= x^3 - (\alpha_1 + \alpha_2 + \alpha_3)x^2 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)x - (\alpha_1\alpha_2\alpha_3), \end{aligned}$$

hence

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 &= 0, \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 &= 1, \\ \alpha_1\alpha_2\alpha_3 &= -1. \end{cases}$$

(b): Now let $t \in \mathbb{R}$ be any real number. Expanding the right hand side of $g_t(x)$ gives

$$g_t(x) = x^3 - Ax^2 + Bx - C,$$

where

$$\begin{aligned}
A &= (\alpha_1 + \alpha_2 + t\alpha_1\alpha_2) + (\alpha_1 + \alpha_3 + t\alpha_1\alpha_3) + (\alpha_2 + \alpha_3 + t\alpha_2\alpha_3) \\
&= (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)\, t \\
&\quad + 2\,\alpha_1 + 2\,\alpha_2 + 2\,\alpha_3 \\
&= (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)\, t \\
&\quad + 2(\alpha_1 + \alpha_2 + \alpha_3) \\
&= (1)t + 2(0) \\
&= t
\end{aligned}$$

and

$$\begin{aligned}
B &= (\alpha_1 + \alpha_2 + t\alpha_1\alpha_2)(\alpha_1 + \alpha_3 + t\alpha_1\alpha_3) \\
&\quad + (\alpha_1 + \alpha_2 + t\alpha_1\alpha_2)(\alpha_2 + \alpha_3 + t\alpha_2\alpha_3) \\
&\quad + (\alpha_1 + \alpha_3 + t\alpha_1\alpha_3)(\alpha_2 + \alpha_3 + t\alpha_2\alpha_3) \\
&= \left({\alpha_1}^2\alpha_2\alpha_3 + \alpha_1{\alpha_2}^2\alpha_3 + \alpha_1\alpha_2{\alpha_3}^2\right) t^2 \\
&\quad + \left({\alpha_1}^2\alpha_2 + {\alpha_1}^2\alpha_3 + \alpha_1{\alpha_2}^2 + 6\,\alpha_1\alpha_2\alpha_3 + \alpha_1{\alpha_3}^2 + {\alpha_2}^2\alpha_3 + \alpha_2{\alpha_3}^2\right) t \\
&\quad + {\alpha_1}^2 + 3\,\alpha_1\alpha_2 + 3\,\alpha_1\alpha_3 + {\alpha_2}^2 + 3\,\alpha_2\alpha_3 + {\alpha_3}^2 \\
&= (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1\alpha_2\alpha_3)t^2 \\
&\quad + [(\alpha_1 + \alpha_2 + \alpha_3)\,(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 3\,\alpha_1\alpha_2\alpha_3]\, t \\
&\quad [(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + (\alpha_1 + \alpha_2 + \alpha_3)] \\
&= (0)(-1)t^2 + [(0)(1) + 3(-1)]t + [(1) + (0)] \\
&= -3t + 1
\end{aligned}$$

and

$$\begin{aligned}
C &= (\alpha_1 + \alpha_2 + t\alpha_1\alpha_2)(\alpha_1 + \alpha_3 + t\alpha_1\alpha_3)(\alpha_2 + \alpha_3 + t\alpha_2\alpha_3) \\
&= {\alpha_1}^2{\alpha_2}^2{\alpha_3}^2 t^3 \\
&\quad + \left(2\,{\alpha_1}^2{\alpha_2}^2\alpha_3 + 2\,{\alpha_1}^2\alpha_2{\alpha_3}^2 + 2\,\alpha_1{\alpha_2}^2{\alpha_3}^2\right) t^2 \\
&\quad + \left({\alpha_1}^2{\alpha_2}^2 + 3\,{\alpha_1}^2\alpha_2\alpha_3 + {\alpha_1}^2{\alpha_3}^2 + 3\,\alpha_1{\alpha_2}^2\alpha_3 + 3\,\alpha_1\alpha_2{\alpha_3}^2 + {\alpha_2}^2{\alpha_3}^2\right) t \\
&\quad + {\alpha_1}^2\alpha_2 + {\alpha_1}^2\alpha_3 + \alpha_1{\alpha_2}^2 + 2\,\alpha_1\alpha_2\alpha_3 + \alpha_1{\alpha_3}^2 + {\alpha_2}^2\alpha_3 + \alpha_2{\alpha_3}^2 \\
&= (\alpha_1\alpha_2\alpha_3)^2 t^3 \\
&\quad + [2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)(\alpha_1\alpha_2\alpha_3)]\, t^2 \\
&\quad + \left[(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)^2 + (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1\alpha_2\alpha_3)\right] t \\
&\quad + [(\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - (\alpha_1\alpha_2\alpha_3)] \\
&= (-1)^2 t^3 + 2(1)(-1)t^2 + [(1)^2 + (0)(-1)]t + [(0)(1) - (-1)] \\
&= t^3 - 2t^2 + t + 1.
\end{aligned}$$

Hence

$$g_t(x) = [x - (\alpha_1 + \alpha_2 + t\alpha_1\alpha_2)] \cdot [x - (\alpha_1 + \alpha_3 + t\alpha_1\alpha_3)] \cdot [x - (\alpha_2 + \alpha_3 + t\alpha_2\alpha_3)]$$

$$= x^3 - (t)x^2 + (-3t+1)t - (t^3 - 2t^2 + t + 1).$$

Remark: I don't know what I was thinking when I assigned this. The computations are extremely tedious. I used a computer to output the latex code. If I did it by hand I would have made 1000 mistakes. It is an unfortunate feature of this subject that the smallest non-trivial examples cannot be done by hand. Therefore the subject often seems more abstract than it really is. Perhaps I should teach the students how to do this on a computer.