Math 461 **Spring 2023**
Homework 3 **Drew Armstrong**

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1. Complex Conjugation. For any complex number $a + ib \in \mathbb{C}$ we define its *complex conjugate* $\alpha^* := a - ib \in \mathbb{C}$.

(a) For any $\alpha \in \mathbb{C}$, show that $\alpha = \alpha^*$ if and only if $\alpha \in \mathbb{R}$.
(b) For any $\alpha, \beta \in \mathbb{C}$ show that $(\alpha+\beta)^* = \alpha^* + \beta^*$ and $(\alpha\beta)^* = \alpha^*\beta^*$.
(c) For any real polynomial $f(x) \in \mathbb{R}[x]$ and complex number $\alpha \in \mathbb{C}$, combine parts (a) and (b) to show that $f(\alpha)^* = f(\alpha^*)$.
(d) For any complex number $\alpha \in \mathbb{C}$ show that the polynomial $(x-\alpha)(x-\alpha^*)$ has real coefficients. [Hint: Show that $\alpha + \alpha^*$ and $\alpha\alpha^*$ are real.]

2. Roots of Unity. Recall Euler's formula

$$e^{it} = \cos t + i \sin t \quad \text{for any real number } t \in \mathbb{R}.$$

Fix an integer $n \geq 1$ and let $\omega = e^{i2\pi/n}$.

(a) For any integer $k \in \mathbb{Z}$, use Euler's formula to show that $(\omega^k)^n = 1$.
(b) For any integers $k, \ell \in \mathbb{Z}$, use Euler's formula to show that

$$\omega^k = \omega^\ell \quad \Longleftrightarrow \quad k - \ell = mn \text{ for some integer } m \in \mathbb{Z}.$$

(c) For any integer $k \in \mathbb{Z}$, use Euler's formula to show that $(\omega^k)^* = \omega^{-k}$.
(d) It follows from (a) and (b) that the polynomial $x^n - 1$ can be factored as

$$x^n - 1 = (x-\omega^0)(x-\omega^1)(x-\omega^2)\cdots(x-\omega^{n-1}).$$

Use this factorization to show that

$$x^n - \alpha^n = (x-\omega^0\alpha)(x-\omega^1\alpha)(x-\omega^2\alpha)\cdots(x-\omega^{n-1}\alpha)$$

for any complex number $\alpha \in \mathbb{C}$. [Hint: Replace x by x/α.]

3. Leibniz' Mistake. Fix a positive real number $a > 0$. In 1702, Leibniz claimed that the polynomial $x^4 + a^4$ cannot be factored over the real numbers. In this problem you will show that Leibniz was wrong.

(a) Let $\lambda = e^{i\pi/4} = (1+i)/\sqrt{2}$. Use Euler's formula to show that $\lambda^2 = i$ and $\lambda^4 = -1$.
(b) Substitute $\alpha = \lambda a$ into Problem 2(d) and use the idea from Problem 1(d) to show that

$$x^4 + a^4 = (x^2 + a\sqrt{2}x + a^2)(x^2 - a\sqrt{2}x + a^2).$$

[It's easy to **check** that this factorization it correct. I want you to **derive** the factorization using properties of complex numbers.]

4. Fifth Roots of Unity. Let $\omega = e^{i2\pi/5}$ so that

$$x^5 - 1 = (x-\omega^0)(x-\omega^1)(x-\omega^2)(x-\omega^3)(x-\omega^4).$$

(a) Use this factorization to show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$. [Hint: Expand the right hand side and compare coefficients.]
(b) Show that $\omega^3 = \omega^{-2}$ and $\omega^4 = \omega^{-1}$, so that $1 + \omega + \omega^2 + \omega^{-2} + \omega^{-1} = 0$. [Hint: 2(b)]
(c) Let $\alpha = \omega + \omega^{-1}$ and use part (b) to show that $\alpha^2 + \alpha - 1 = 0$.
(d) Solve the quadratic equation in (c) to get an explicit formula for $\cos(2\pi/5)$. [Hint: We know from Euler's formula or Problem 1(d) that $\omega + \omega^{-1} = 2\cos(2\pi/5)$.]