Today: Impossible Constructions.

Wed: Gauss-Wantzel Theorem.

Recall:

- A point $(a,b) \in \mathbb{R}^2$ is constructible if it can be obtained from $(0,0)$ & $(1,0)$ via (Euclidean) compass & straightedge.

- Let $\mathbb{Q}_{sqrt}$ be set of real numbers obtainable from 0 & 1 by repeated operations of the form $+, -, \times, \div, \sqrt{\cdot}$.

  This is a field $\mathbb{Q} \subseteq \mathbb{Q}_{sqrt} \subseteq \mathbb{R}$.

e.g. $$2 + \sqrt{\frac{1+\sqrt{3+\sqrt{5}}}{2}} \in \mathbb{Q}_{sqrt}.$$

Theorem (Descartes 1637, officially: Wantzel 1837):

point (a,b) constructible $\iff$ $a, b \in \mathbb{Q}_{sqrt}$.

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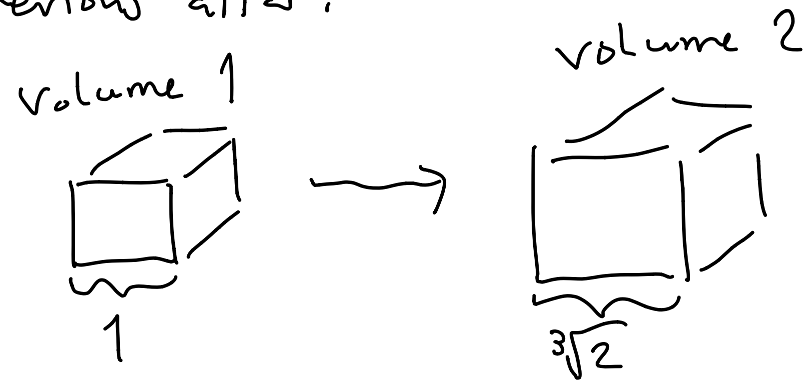
Today we will use this to prove the impossibility of 3 classical problems:

(1) The "Delian Problem."

Oracle at Delos: To stop the plague construct new altar of twice size of previous altar.



Required to use straightedge & compass.

Modern Translation:

Given points $(0,0)$ & $(1,0)$ construct the point $(\sqrt[3]{2}, 0)$, i.e.,

$$\sqrt[3]{2} \in \mathbb{Q}_{sqrt} \ ?$$

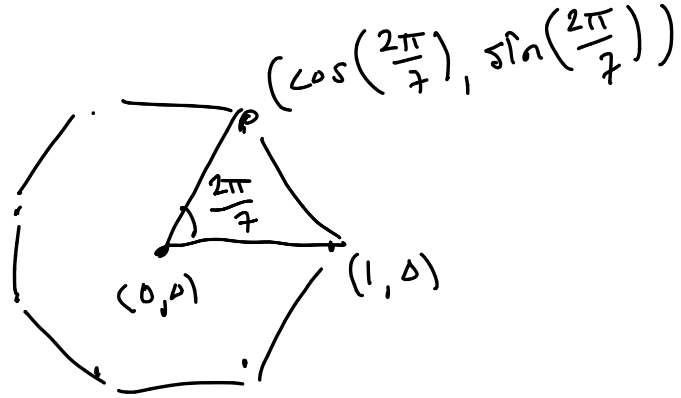
(2) Angle Trisection.

Given angle θ, construct angle $\theta/3$:

Algebraically:

$$\cos\theta \in \mathbb{Q}_{sqrt} \overset{?}{\Longrightarrow} \cos\frac{\theta}{3} \in \mathbb{Q}_{sqrt}.$$

③ Construct Regular Heptagon.



$$\cos\left(\frac{2\pi}{7}\right) \in \mathbb{Q}_{sqrt} \quad ?$$

[Note: $\cos\theta \in \mathbb{Q}_{sqrt} \Longleftrightarrow \sin\theta \in \mathbb{Q}_{sqrt}$.].

We will prove that ①, ②, ③ are impossible by showing that

① $\sqrt[3]{2} \notin \mathbb{Q}_{sqrt}$

② $\cos\left(\frac{2\pi}{3}\right) \in \mathbb{Q}_{sqrt}$

But $\cos\left(\frac{2\pi}{9}\right) \notin \mathbb{Q}_{sqrt}$.

③ $\cos\left(\frac{2\pi}{7}\right) \notin \mathbb{Q}_{sqrt}$.

Amazingly, we can prove all 3 at the same time with the following result.

Theorem: Let $f(x) \in \mathbb{Q}[x]$ have degree 3. Then

$f(x)$ has a root in $\mathbb{Q}_{sqrt}$ $\Longrightarrow$ $f(x)$ has a root in $\mathbb{Q}$.

Equivalently,

$f(x)$ has no root in $\mathbb{Q}$ $\Longrightarrow$ $f(x)$ has no root in $\mathbb{Q}_{sqrt}$.

Before proving this let's see how to use it.

(1) Let $f(x) = x^3 - 2 \in \mathbb{Q}[x]$
and $\alpha = \sqrt[3]{2} \in \mathbb{R}$.

Note $f(\alpha) = 0$. But $f(x)$ has <u>no</u> root in $\mathbb{Q}$, hence has <u>no</u> root in $\mathbb{Q}$sqrt, hence $\alpha \notin \mathbb{Q}$sqrt.

(2) Let $\alpha = \cos\left(\frac{2\pi}{9}\right)$. Triple angle identity:

$$4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3} - \cos\theta = 0.$$

Let $\theta = 2\pi/3$:

$$4\cos^3\left(\frac{2\pi}{9}\right) - 3\cos\left(\frac{2\pi}{9}\right) - \cos\left(\frac{2\pi}{3}\right) = 0$$

$$4\alpha^3 - 3\alpha + \frac{1}{2} = 0.$$

$$8\alpha^3 - 6\alpha + 1 = 0.$$

Let $f(x) = 8x^3 - 6x + 1 \in \mathbb{Q}[x]$.

since $f(\alpha) = 0$ and $f(x)$ has no root in $\mathbb{Q}$ (check), we see that

$$\alpha \notin \mathbb{Q}\text{sqrt}.$$

(3) Let $\alpha = \cos\left(\frac{2\pi}{7}\right)$. You showed on HW4 that

$$(2\alpha)^3 + (2\alpha)^2 - 2(2\alpha) - 1 = 0$$

Since $f(x) = x^3 + x^2 - 2x - 1 \in \mathbb{Q}[x]$ has no root in $\mathbb{Q}$, we conclude that $2\alpha \notin \mathbb{Q}\text{sqrt}$, which Implies $\alpha \notin \mathbb{Q}\text{sqrt}$.

[because $\alpha \in \mathbb{Q}\text{sqrt} \Rightarrow 2\alpha \in \mathbb{Q}\text{sqrt}$.]

DONE ✓

So let's prove the theorem.

Proof: Let $f(x) \in \mathbb{Q}[x]$ have degree 3, and suppose that $f(x) = 0$ for some $\alpha \in \mathbb{Q}_{sqrt}$. We will show that f has <u>some</u> root in $\mathbb{Q}$.

By definition of $\mathbb{Q}_{sqrt}$, there exists a chain of "quadratic field extensions":

$$\alpha \in \mathbb{F}_l \supseteq \mathbb{F}_{l-1} \supseteq \cdots \supseteq \mathbb{F}_2 \supseteq \mathbb{F}_1 \supseteq \mathbb{Q}$$

where $\mathbb{F}_k$ is obtained from $\mathbb{F}_{k-1}$ by "adjoining" the square root of some number, i.e.,

$$\mathbb{F}_k = \mathbb{F}_{k-1}(L_k) \text{ where}$$

$$L_k \in \mathbb{F}_k - \mathbb{F}_{k-1} \;\&\; L_k^2 \in \mathbb{F}_{k-1}.$$

[Example: To get $\alpha = 1 + \sqrt{\frac{1+\sqrt{2}}{3}}$,

$$\begin{array}{ccccc} \mathbb{Q} & \subseteq & \mathbb{Q}(\sqrt{2}) & \subseteq & \mathbb{Q}(\sqrt{2})\left(\sqrt{\frac{1+\sqrt{2}}{3}}\right) \\ 1 & & \frac{1+\sqrt{2}}{3} & & 1+\sqrt{\frac{1+\sqrt{2}}{3}} \end{array}$$

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Using the idea that a QFE $\mathbb{F}(\iota) \supseteq \mathbb{F}$ behaves like $\mathbb{C} \supseteq \mathbb{R}$ you proved on HW5 that for any $f(x) \in \mathbb{F}[x]$ of deg 3,

$$\begin{array}{ccc} f(x) \text{ has root} & \Longrightarrow & f(x) \text{ has root} \\ \text{in } \mathbb{F}(\iota) & & \text{in } \mathbb{F}. \end{array}$$

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For our polynomial $f(x) \in \mathbb{Q}[x]$ of degree 3, note that $f(x) \in \mathbb{F}_k[x]$ for all k because $\mathbb{F}_k \supseteq \mathbb{Q}$.

By repeatedly applying the above lemma:

f has a root in $\mathbb{F}_\ell$

$\Rightarrow$ f has a root in $\mathbb{F}_{\ell-1}$

$\vdots$

$\Rightarrow$ f has a root in $\mathbb{F}_1$

$\Rightarrow$ f has a root in $\mathbb{Q}$.

QED.

That's it.

Discussion:

Do you like this proof?

What would the Greeks think?

This doesn't even look like geometry!

Sometimes we can only make progress in mathematics by

changing the rules.

But this proof only works for polynomials of degree 3.

More generally: Suppose $\alpha \in \mathbb{R}$ satisfies $p(\alpha) = 0$ for some prime polynomial $p(x) \in \mathbb{Q}[x]$.

Then it is true that

$$\alpha \in \mathbb{Q}_{\text{sqrt}} \iff \deg(p) \text{ is a power of } 2$$

But this is <u>much</u> harder to prove.

This (and related problems) led to a complete transformation of "algebra" in the years 1830 – 1930.

I will give a brief survey on Wednesday. [See my book "Algebra: 1830-1930" for the full story. Coming Soon!]