Hello Again!

Writing Project due today.

HW4: Soon. Will be due Apr 10.

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Last time...

Abstract Vector space is

Set $V$ (of vectors)

Field $\mathbb{F}$ (of scalars)

Two operations:

$u, v \in V \Rightarrow u+v \in V$ vector addition

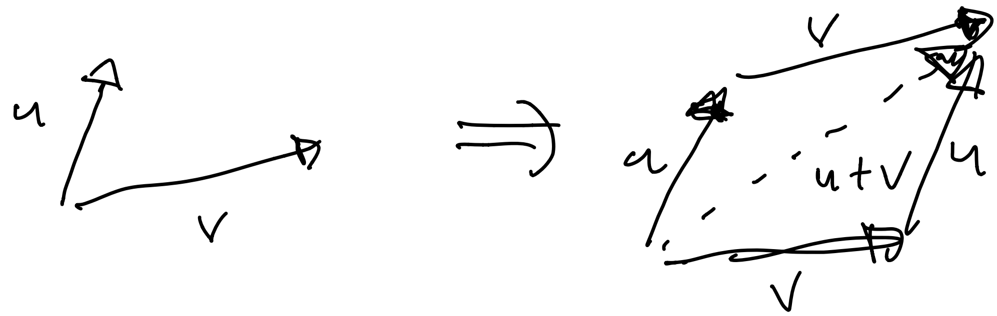
$a \in \mathbb{F}, u \in V \Rightarrow au \in V$ scalar multiplication

Warning: There is no "vector multiplication".

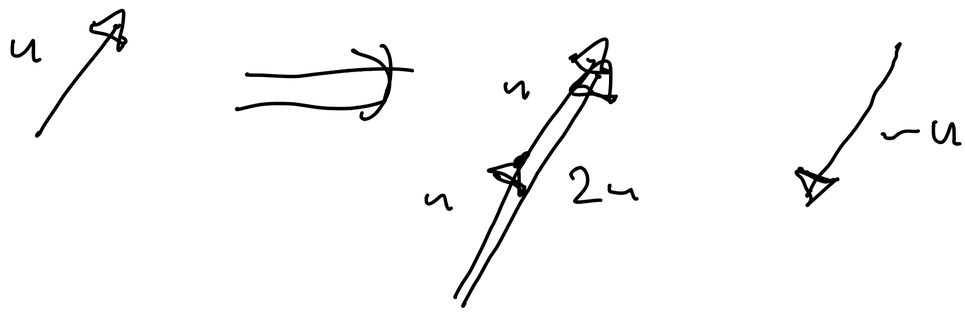
$u, v \in V \not\Rightarrow u \cdot v \in V$

Makes no sense, except in very special cases.

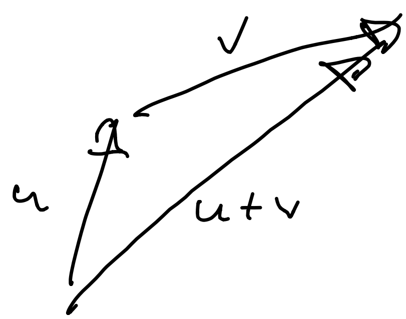
Prototype: Arithmetic of directed line segments.




$u + v = v + u.$




Let $\|u\|$ be the length of the directed line segment. Have a "triangle inequality"




$$\|u+v\| \leq \|u\| + \|v\|$$

Idea goes back to Isaac Newton to describe forces. Modern definition of "vector space":

Grassmann 1840s - 1850s

Peano 1880s

Weyl 1920s Quantum Physics

Why do we care?

Because $\mathbb{C}$ form a vector space over the field $\mathbb{R}$.

Operations: Given $\alpha, \beta \in \mathbb{C}$, $a \in \mathbb{R}$ we have "vector addition"

$$\alpha + \beta$$

and "scalar multiplication"

$$a\alpha$$

Easy: Check that vector space axioms are satisfied.

Who cares? For each $\alpha \in \mathbb{C}$ we will define a function

$$f_\alpha : \mathbb{C} \to \mathbb{C}$$

How? Define for all $\beta \in \mathbb{C}$,

$$\boxed{f_\alpha(\beta) := \alpha\beta}$$

This is not just any kind of function. It is a $\mathbb{R}$-linear function.

Check: for all "vectors" $\beta, \gamma \in \mathbb{C}$ and all "scalars" $a, b \in \mathbb{R}$ we have

$$
\begin{aligned}
f_\alpha(a\beta + b\gamma) &= \alpha(a\beta + b\gamma) \\
&= a(\alpha\beta) + b(\alpha\gamma) \\
&= a f_\alpha(\beta) + b f_\alpha(\gamma). \quad \checkmark
\end{aligned}
$$

It follows that $f_\alpha$ can be represented as a matrix with real entries.

How? Standard Basis of $\mathbb{C}$?

$$
\begin{aligned}
a + bi &\longleftrightarrow (a, b) \\
1 &\longleftarrow (1, 0) \\
i &\longleftarrow (0, 1)
\end{aligned}
$$

Let $\alpha = a + bi$. What does $f_\alpha$ do to the standard basis?

$$f_\alpha(1) = \alpha 1 = \alpha = a+bi \qquad (a,b)$$

$$f_\alpha(i) = \alpha i = (a+bi)i = -b + ai \qquad (-b,a)$$

The matrix is

$$[f_\alpha] = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Summary: Identify each complex number with 2×2 real matrix:

$$\underbrace{a+bi}_{\alpha} \longleftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = [f_\alpha]$$

In modern terms we could <u>define</u> a "complex number" as such a matrix.

Key fact:

$$\boxed{[f_\alpha]\,[f_\beta] = [f_{\alpha\beta}]}$$

matrix multiplication

multiplication of complex numbers

Examples:

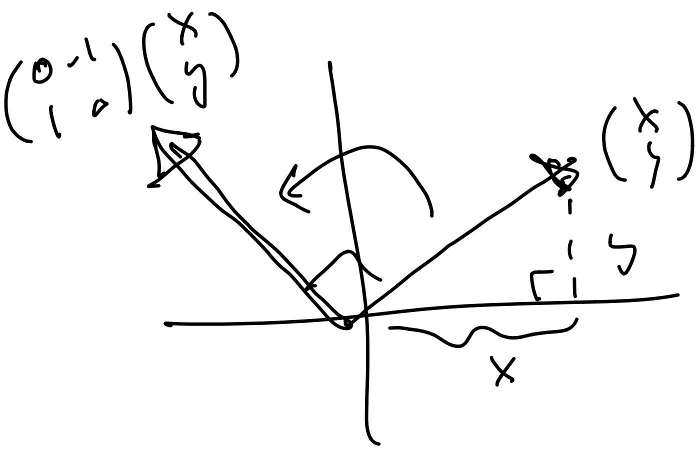
$\alpha = 1 \Rightarrow \alpha = 1 + 0i \rightsquigarrow \begin{pmatrix} 1 & -0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

This is the identity matrix, corresponding to the identity function.
DO NOTHING FUNCTION.

$\alpha = i \Rightarrow \alpha = 0 + 1i \rightsquigarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

What does it do? We saw last time that this matrix is "rotate c.c.w. by 90°" function.




This is what the number $i$ "really is."

$$\alpha = e^{i\theta} \implies \alpha = \cos\theta + i\sin\theta$$

$$\rightsquigarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

This is the function that rotates ccw by angle $\theta$.

$$\alpha = re^{i\theta} \longrightarrow \alpha = r\cos\theta + i\, r\sin\theta$$

$$\rightsquigarrow \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}$$

$$= r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

actually order doesn't matter

This function rotates by $\theta$ then scales (amplifies) by factor $r$.

"Amplitwist"

Summary: We can think of complex numbers as "amplitwist functions".

I'll ask to to explore these definitions on next homework.

OKAY

Moving On. Back to the story.

Every polynomial we have studied so far splits over the complex numbers.

e.g. $x^3 - 1 = (x-1)\left(x - \frac{-1+i\sqrt{3}}{2}\right)\left(x - \frac{-1-i\sqrt{3}}{2}\right)$.

By 1600s, it was generally believed that this is true for every polynomial.

i.e. $\mathbb{C}$ contains all the roots of every polynomial.

However, no one knew how to prove this. It stumped generations of mathematicians.