Math 461 **Spring 2020**
Homework 4 **Drew Armstrong**

Problem 1. Complex Numbers as Real 2×2 Matrices. For any complex number $\alpha = a + bi \in \mathbb{C}$ with $a, b \in \mathbb{R}$ we define the following matrix:

$$M_\alpha := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

(a) Check that for all $r \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ we have $M_{(r\alpha)} = rM_\alpha$.
(b) Check that for all $\alpha, \beta \in \mathbb{C}$ we have $M_{\alpha+\beta} = M_\alpha + M_\beta$ and $M_{\alpha\beta} = M_\alpha M_\beta$.
(c) Check that for all $\alpha \in \mathbb{C}$ we have $\det(M_\alpha) = |\alpha|^2$.
(d) Check that for all $\alpha \in \mathbb{C}$ we have $(M_\alpha)^* = M_{(\alpha^*)}$, where the star operation denotes the transpose matrix and the complex conjugate, respectively.

Problem 2. Greatest Common Divisor. Let $a, b \in \mathbb{Z}$ with $d = \gcd(a, b)$. Since d is a common divisor of a and b we must have $a = da'$ and $b = db'$ for some integers $a', b' \in \mathbb{Z}$. In this case, prove that the numbers a', b' are *coprime*:

$$\gcd(a', b') = 1.$$

[HInt: From Bézout's Identity we know that $ax + by = d$ for some (non-unique) integers $x, y \in \mathbb{Z}$. Use this to show that any common divisor $e|a'$ and $e|b'$ must satisfy $e|1$.]

Problem 3. Euclid's Lemma. For all integers $a, b, c \in \mathbb{Z}$, prove that

$$(a|bc \text{ and } \gcd(a, b) = 1) \quad \Rightarrow \quad a|c.$$

[Hint: If $\gcd(a, b) = 1$ then from Bézout's Identity there exist some (non-unique) integers $x, y \in \mathbb{Z}$ satisfying $ax + by = 1$. Multiply both sides by c to get $acx + bcy = c$. Now what?]

Problem 4. Rational Root Test. Let $f(x) = c_n x^n + \cdots + c_1 x + c_0 \in \mathbb{Z}[x]$ be a polynomial of degree n with integer coefficients. Suppose that $f(x)$ has a rational root $a/b \in \mathbb{Q}$ in lowest terms, i.e., with $\gcd(a, b) = 1$. In this case, prove that we must have

$$a|c_0 \quad \text{and} \quad b|c_n.$$

[HInt: Suppose that $f(a/b) = 0$. Multiply both sides by b^n and then use Euclid's Lemma.]

Problem 5. The Regular 7-Gon. Let $\omega = e^{2\pi i/7}$ and $\alpha = \omega + \omega^{-1} = 2\cos(2\pi i/7)$.

(a) Combine the numbers $1, \alpha, \alpha^2, \alpha^3$ to find some polynomial $f(x) \in \mathbb{Z}[x]$ of degree 3 satisfying $f(\alpha) = 0$. [Hint: Use the fact that $\omega^3 + \omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} + \omega^{-3} = 0$.]
(b) Use Problem 4 to show that your polynomial $f(x)$ from part (a) has no rational roots.
(c) Use part (b) to prove that the real number $\cos(2\pi/7) \in \mathbb{R}$ is **irrational**.

Problem 6. Conjugation of Complex Polynomials. For any polynomial $f(x) = \sum_{k\geq 0} a_k x^k \in \mathbb{C}[x]$ with complex coefficients, we define the *conjugate polynomial* as follows:

$$f^*(x) := \sum_{k\geq 0} a_k^* x^k.$$

(a) We can think of $\mathbb{R}[x] \subseteq \mathbb{C}[x]$ as a subring. For all $f(x) \in \mathbb{C}[x]$, prove that

$$f(x) \in \mathbb{R}[x] \quad \Leftrightarrow \quad f^*(x) = f(x).$$

(b) For all $f(x), g(x) \in \mathbb{C}[x]$, prove $(f + g)^*(x) = f^*(x) + g^*(x)$ and $(fg)^*(x) = f^*(x)g^*(x)$.
(c) For all $f(x) \in \mathbb{C}[x]$ use (a),(b) to prove that $f(x) + f^*(x) \in \mathbb{R}[x]$ and $f(x)f^*(x) \in \mathbb{R}[x]$.