Math 461 **Spring 2020**
Exam 1 (Fri Feb 21) **Drew Armstrong**

There are 4 pages, each with 6 points, for a total of 24 points. If copying is detected then the exam will receive zero points. No electronic devices are allowed.

Problem 1. One can check that $x = 2$ is a root of $f(x) = x^3 - 4x^2 + 6x - 4 \in \mathbb{Q}[x]$.

(a) Find a polynomial $g(x) \in \mathbb{Q}[x]$ such that $f(x) = (x - 2)g(x)$.

$$\begin{array}{r|l} & x^2 - 2x + 2 \\ x - 2 & x^3 - 4x^2 + 6x - 4 \\ & -x^3 + 2x^2 \\ \hline & -2x^2 + 6x \\ & 2x^2 - 4x \\ \hline & 2x - 4 \\ & -2x + 4 \\ \hline & 0 \end{array}$$

(b) Use the quadratic formula to find all roots of $g(x)$.

From (a) we have $g(x) = x^2 - 2x + 2$. The quadratic formula gives

$$x = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

(c) Use your result to express $f(x)$ in the form $(x - r)(x - s)(x - t)$ for some $r, s, t \in \mathbb{C}$.

$$f(x) = (x - 2)(x^2 - 2x + 2) = (x - 2)(x - 1 - i)(x - 1 + i)$$

Problem 2. Let $\mathbb{F}$ be a field. Suppose that the polynomial $x^2 + ax + b \in \mathbb{F}[x]$ can be factored as $f(x) = (x - r)(x - s)$ for some numbers $r, s \in \mathbb{F}$.

(a) Expand the right hand side to find expressions for a, b in terms of r, s.

We have $x^2 + ax + b = (x - r)(x - s) = x^2 - (r + s)x + rs$. Comparing coefficients gives

$$\begin{cases} a &= -(r + s), \\ b &= rs. \end{cases}$$

(b) Use (a) to find an expression for $(r - s)^2$ in terms of a, b.

First observe that $a^2 = (r + s)^2 = r^2 + 2rs + s^2$. Then

$$\begin{aligned} (r - s)^2 &= r^2 - 2rs + s^2 \\ &= r^2 + 2rs + s^2 - 4rs \\ &= (r + s)^2 - 4rs \\ &= a^2 - 4b. \end{aligned}$$

(c) Assuing that $r \neq 0$ and $s \neq 0$, use (a) to find an expression for $1/r^2 + 1/s^2$ in terms of a, b. [Hint: First find a common denominator.]

First observe that $r^2 + s^2 = (r+s)^2 - 2rs = a^2 - 2b$. Then

$$\begin{aligned} 1/r^2 + 1/s^2 &= (s^2 + r^2)/(r^2 s^2) \\ &= (a^2 - 2b)/b^2. \end{aligned}$$

Problem 3. Consider the ring of polynomials $\mathbb{R}[x]$ with coefficients in $\mathbb{R}$.

(a) State the definition of a prime polynomial in $\mathbb{R}[x]$.

We say that $p(x) \in \mathbb{R}[x]$ is prime when:

- $\deg(p) \geq 1$,
- $p(x) = f(x)g(x)$ implies $\deg(f) = 0$ or $\deg(g) = 0$.

(b) Prove that any polynomial $f(x) \in \mathbb{R}[x]$ of degree 1 has a root in $\mathbb{R}$.

If $\deg(f) = 1$ then we can write $f(x) = ax + b$ for some $a, b \in \mathbb{R}$ with $a \neq 0$. Then we observe that $-b/a \in \mathbb{R}$ is a root:

$$f(-b/a) = a(-b/a) + b = -b + b = 0.$$

(c) You may assume that the polynomial $p(x) = x^2 + 1 \in \mathbb{R}[x]$ has no root in $\mathbb{R}$. Use this information to prove that $p(x)$ is prime in $\mathbb{R}[x]$.

Assume for contradiction that $p(x)$ is not prime. By definition this means that we can write $p(x) = f(x)g(x)$ for some polynomials $f(x), g(x) \in \mathbb{R}[x]$ with $\deg(f) \geq 1$ and $\deg(g) = 1$. Since

$$2 = \deg(p) = \deg(fg) = \deg(f) + \deg(g),$$

this implies that $\deg(f) = \deg(g) = 1$. In particular, from part (b) we know that $f(x)$ has a root, say $r \in \mathbb{R}$. But then we have

$$p(r) = f(r)g(r) = 0 \cdot g(r) = 0,$$

contradicting the fact that $p(x)$ has no root in $\mathbb{R}$.

Problem 4. Let $f(x) \in \mathbb{R}[x]$ have real coefficients and let $i \in \mathbb{C}$ be the imaginary unit.

(a) If $f(x) = (x^2 + 1)g(x)$ for some $g(x) \in \mathbb{R}[x]$ show that $f(i) = 0$.

$$f(i) = (i^2 + 1)g(i) = (-1 + 1)g(i) = 0 \cdot g(i) = 0$$

(b) If $r(x) \in \mathbb{R}[x]$ has degree 1 show that $r(i) \neq 0$. [Hint: You can assume that $i \notin \mathbb{R}$.]

If $\deg(r) = 1$ then we can write $r(x) = ax + b$ for some $a, b \in \mathbb{R}$ with $a \neq 0$. If $r(i) = 0$ then we obtain the contradiction

$$\begin{aligned} r(i) &= 0 \\ ai + b &= 0 \\ i &= -b/a \in \mathbb{R}. \end{aligned}$$

(c) If $f(i) = 0$ show that $f(x) = (x^2 + 1)g(x)$ for some $g(x) \in \mathbb{R}[x]$. [Hint: Divide $f(x)$ by $x^2 + 1$ in the ring $\mathbb{R}[x]$. Show that the remainder is the zero polynomial.]

Suppose that $f(i) = 0$ for some $f(x) \in \mathbb{R}[x]$. Divide $f(x)$ by $x^2 + 1$ to obtain polynomials $q(x), r(x) \in \mathbb{R}[x]$ satisfying

$$\begin{cases} f(x) = (x^2 + 1)q(x) + r(x), \\ \deg(r) < 2. \end{cases}$$

Since $f(i) = 0$ it follows that

$$0 = f(i) = (i^2 + 1)q(i) + r(i) = 0g(i) + r(i) = r(i).$$

I claim that this implies $r(x) = 0(x)$, and hence $f(x) = (x^2 + 1)q(x)$ as desired. To prove this, suppose for contradiction that $r(x) \neq 0(x)$. Then there are two cases:

- If $\deg(r) = 0$ then $r(x) = c \in \mathbb{F}$ is a nonzero constant, hence $r(i) = c \neq 0$.
- If $\deg(r) = 1$ then from part (b) we also have $r(i) \neq 0$.

In either case we obtain a contradiction.

Remark: This is an extension of Descartes' Factor Theorem. The original version says that for any polynomial $g(x) \in \mathbb{C}[x]$ we have

$$g(i) = 0 \text{ in the field } \mathbb{C} \quad \Leftrightarrow \quad (x - i) | g(x) \text{ in the ring } \mathbb{C}[x].$$

Our new version says that for any polynomial $f(x) \in \mathbb{R}[x]$ we have

$$f(i) = 0 \text{ in the field } \mathbb{C} \quad \Leftrightarrow \quad (x^2 + 1) | f(x) \text{ in the ring } \mathbb{R}[x].$$

More generally, for any pair of fields $\mathbb{F} \subseteq \mathbb{E}$ and for any element $\alpha \in \mathbb{E}$, we will see that there exists some polynomial $m_\alpha(x) \in \mathbb{F}[x]$ (which is either prime or zero) such that for all $f(x) \in \mathbb{F}[x]$ we have

$$f(\alpha) = 0 \text{ in the field } \mathbb{E} \quad \Leftrightarrow \quad m_\alpha(x) | f(x) \text{ in the ring } \mathbb{F}[x].$$

We call $m_\alpha(x)$ the *minimal polynomial for α over $\mathbb{F}$*. Thus $x^2 + 1$ is the minimal polynomial for i over $\mathbb{R}$ (also over $\mathbb{Q}$).