HW4 Hints:

Problem 6, let $f(x) = a_2 x^2 + a_1 x + a_0$ and $g(x) = b_2 x^2 + b_1 x + b_0$ where $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{C}$. We define "conjugate polynomials"

$$f^*(x) = a_2^* x^2 + a_1^* x + a_0^*$$
$$g^*(x) = b_2^* x^2 + b_1^* x + b_0^*$$

Goal: Show that

$$f^*(x) + g^*(x) = (f+g)^*(x)$$

Proof: By definition we have

$$(f+g)(x) := (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0)$$

$$(f+g)^*(x) := (a_2+b_2)^* x^2 + (a_1+b_1)^* x + (a_0+b_0)^*$$

$$= (a_2^*+b_2^*)x^2 + (a_1^*+b_1^*)x + (a_0^*+b_0^*)$$

$$= (a_2^* x^2 + a_1^* x + a_0^*) + (b_2^* x^2 + b_1^* x + b_0^*)$$

$$=: f^*(x) + g^*(x) \quad \checkmark$$

Same works to show $(f+g)^*(x)$
$= f^*(x) + g^*(x)$ for any $f(x), g(x) \in \mathbb{C}[x]$.

Ultimate goal: If we can show that $(fg)^*(x) = f^*(x)g^*(x)$ then we see that

$$f(x)\,f^*(x) = (ff^*)(x)$$

$$\begin{aligned}(ff^*)^*(x) &= (f^*f^{**})(x)\\ &= (f^*f)(x)\\ &= (ff^*)(x)\end{aligned}$$

So, $f(x)\,f^*(x)$ is equal to its own conjugate.

What does that mean?

Hint: See HW 4.6(a).

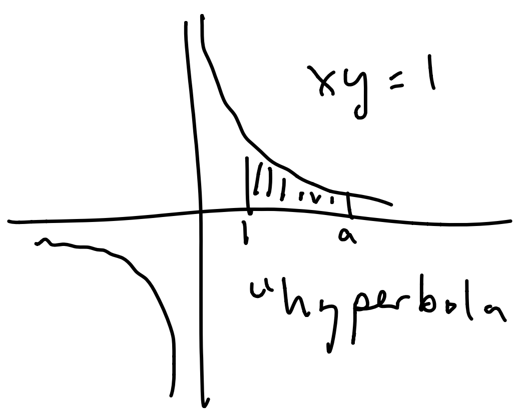
Today: Leibniz' Mistake.

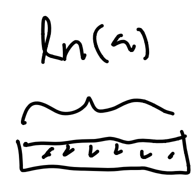
In 1702, Leibniz developed the method of partial fractions to integrate rational functions.

If $g(x)$ = product of degrees 1 & 2 with real coefficients, he realized that for any $f(x)$,

$$\int \frac{f(x)}{g(x)}\,dx = \text{"can be reduced to the quadrature of the hyperbola \& the circle"}$$

i.e. ln & arctan





quadrature has side length ln(a).

But, he claimed that the integral

$$\int \frac{1}{x^4+a^4}\,dx$$

cannot be expressed in these terms, i.e., he claimed that $x^4+a^4 \in \mathbb{R}[x]$ cannot be factored.

Well, he was wrong.

Why did he get confused?

If $f(x) \in \mathbb{F}[x]$ has degree 2 or 3 then we know that

$f(x)$ is prime in $\mathbb{F}[x]$ $\Longleftrightarrow$ $f(x)$ has no root in $\mathbb{F}$.

Sketch: If $f(a)=0$, $a \in \mathbb{F}$ then

$f(x) = (x-a)\,g(x)$ is not prime.

Conversely, suppose $f(x)$ has degree 2 or 3, and is not prime:

$$f(x) = g(x)\,h(x) \qquad \deg(g), \deg(h) \geq 1$$

Then $\deg(g) + \deg(h) = \deg(f) = 2$ or 3.

$\Rightarrow \deg(g) = 1$ W.L.O.G.
without loss of generality.

But then $g(x)$ [and hence $f(x)$] has a root in $\mathbb{F}$. ✓

But this doesn't work for degree 4 & above!

Example: The polynomial

$$(x^2+1)(x^2+x+1) = x^4 + x^3 + 2x^2 + x + 1$$

is not prime in $\mathbb{R}[x]$, but it has no real roots because x^2+1 & x^2+x+1 have no real roots.

"Primality Testing is Hard"

Example: Is 1,532,171 prime?

Example: Is $x^4 + a^4 \in \mathbb{R}[x]$ prime?

Leibniz (temporarily) thought that it is. But we will show that it's not.

Theorem: Any nonzero complex # has exactly n complex nth roots.

Proof: Let $\alpha = re^{i\theta}$ in polar form. and let r' be the positive real nth root of r. (Exists by Intermediate Value Theorem.) Then I claim

$$\alpha' := r' e^{i\theta/n}$$

is an nth root of α.

Indeed, we have

$$(\alpha')^n = \left(r' e^{i\theta/n}\right)^n$$
$$= (r')^n \left(e^{i\theta/n}\right)^n$$
$$= r e^{i\theta} = \alpha \quad \checkmark$$

Now let $\omega = e^{2\pi i/n}$. Then I claim that

$$\alpha', \alpha'\omega, \alpha'\omega^2, \ldots, \alpha'\omega^{n-1}$$

are distinct & give all the nth roots of α.

Indeed: They are nth roots of α:

$$(\alpha'\omega^k)^n = (\alpha')^n (\omega^n)^k$$
$$= \alpha (1)^k = \alpha.$$

They are distinct because

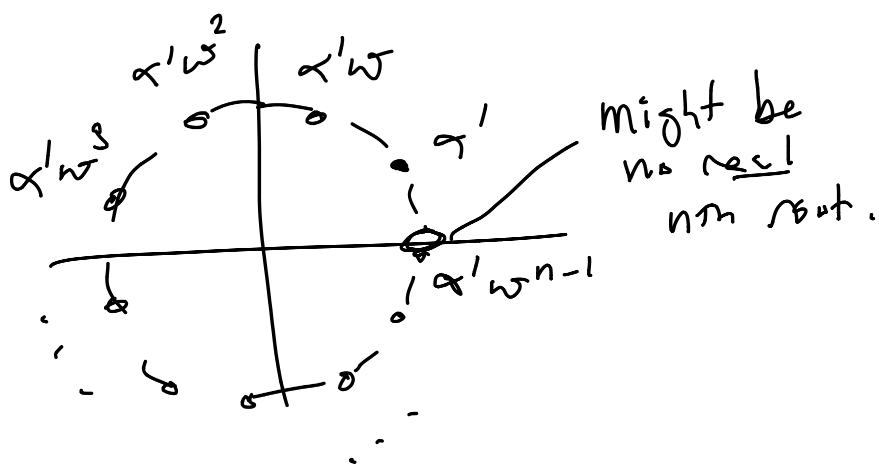
$$\cancel{\alpha'}\omega^k = \cancel{\alpha'}\omega^l$$
$$\omega^k = \omega^l$$

$\Rightarrow k = l$ because $1 \leq k, l < n$.

We cant have any more nth roots because $x^n - \alpha \in \mathbb{C}[x]$ has at most n roots in the field $\mathbb{C}$.

$\square$

Picture: The nth roots of α form a regular n-gon in $\mathbb{C}$ centered at 0.



To find roots of $x^4 + a^4$ we need to solve

$$x^4 + a^4 = 0$$
$$x^4 = -a^4.$$

i.e., we need the 4th roots of $-a^4$, where $a > 0$ is real.

Polar form $\alpha = -a^4$

$$= a^4(-1)$$

$$= a^4 e^{i\pi}$$

"Principal" 4th root

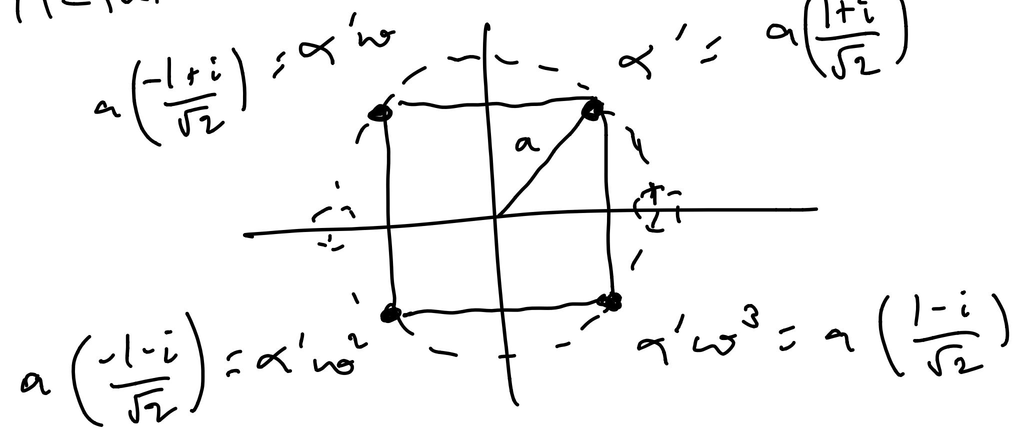
$$\alpha' = a e^{i\pi/4}.$$

All the 4th roots are $\left(\omega = e^{2\pi i/4} = e^{i\pi/2}\right)$

$$\alpha', \quad \alpha'\omega, \quad \alpha'\omega^2, \quad \alpha'\omega^3$$

$$a e^{i\pi/4}, \quad a e^{i\pi 3/4}, \quad a e^{i\pi 5/4}, \quad a e^{i\pi 7/4}$$

Picture:



This gives us a real factorization of $x^4 + 9$

$\vdots$

NEXT TIME!