HW4 due on Fri via email.

Topic: F.T.A.

Example: $$\int \frac{1}{(x+1)^2(x^2+1)}\,dx = ?$$

To begin, an example from $\mathbb{Z}$.

Want to expand $5/12$.

$$12 = 2^2 \cdot 3.$$

repeated prime

First $12 = 4 \cdot 3$

$$1 = \gcd(4,3)$$

$$1 = 4x + 3y \quad \text{for some } x, y \in \mathbb{Z}$$

$$1 = 4(1) + 3(-1)$$

$$\frac{1}{12} = \frac{\cancel{4}(1)}{\cancel{12}} + \frac{\cancel{3}(-1)}{\cancel{12}} = \frac{1}{3} + \frac{-1}{4}$$

$$\frac{5}{12} = \frac{5}{3} + \frac{-5}{4}.$$ not prime!

Standard form:

$$\frac{5}{3} = \frac{1\cdot 3 + 2}{3} = 1 + \frac{2}{3} \quad ✓$$

negative quotient OK.

$$\frac{-5}{4} = \frac{(-3)\cdot 2 + 1}{4}$$

$$= \frac{-3}{2} + \frac{1}{4}$$

$$= \frac{(-2)\cdot 2 + 1}{2} + \frac{1}{4}$$

$$= -2 + \frac{1}{2} + \frac{1}{4} \quad ✓$$

Generally:

$$\frac{a}{p^e} = c + \frac{r_1}{p} + \frac{r_2}{p^2} + \cdots + \frac{r_e}{p^e}$$

where $0 \leq r_i < p$ for all i.

In summary:

$$\frac{5}{12} = \cancel{1} + \frac{2}{3} + (\cancel{-2}) + \frac{1}{2} + \frac{1}{4}$$

$$= -1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{4}$$

The standard form is UNIQUE.

Theorem (Partial Fractions):

Let (R, N) be Euclidean domain.

For any $a, b \in R$, $b \neq 0$, we want to expand a/b into partial fractions.

First, suppose

$$b = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

unique prime factorization.

Then I claim that there exist some elements $c, r_{ij} \in R$

satisfying

$$\left\{\begin{array}{l} \dfrac{a}{b} = c + \displaystyle\sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{r_{ij}}{p_i^{\,j}} \\[2em] \forall i,j,\ r_{ij} = 0 \text{ or } N(r_{ij}) < N(p_i) \end{array}\right.$$

[Often these c, r_{ij} are unique, but we don't need this so we won't prove it.]

<u>Proof</u>: First observe that

$$1 = \gcd\left(p_2^{e_2} \cdots p_k^{e_k},\ p_1^{e_1}\right)$$

Bézout: $\exists\ c_1, c \in R$ with

$$1 = c_1\, p_2^{e_2} \cdots p_k^{e_k} + c\, p_1^{e_1}$$

Now divide by b to get
and multiply by a

$$\frac{a}{b} = ac_1 \frac{\cancel{p_2^{e_2} \cdots p_k^{e_k}}}{p_1^{e_1} \cancel{\cdots p_k^{e_k}}} + ac \frac{\cancel{p_1^{e_1}}}{\cancel{p_1^{e_1}} \cdots p_k^{e_k}}$$

$$= \frac{ac_1}{p_1^{e_1}} + \frac{ac}{p_2^{e_2} \cdots p_k^{e_k}}$$

induction

$$= \frac{a_1}{p_1^{e_1}} + \frac{a_2}{p_2^{e_2}} + \cdots + \frac{a_k}{p_k^{e_k}}$$

for some $a_1, a_2, \ldots, a_k \in R$.

Finally, we put each fraction a/p^e in standard form by repeatedly dividing numerator by p:

$$\frac{a}{p^e} = \frac{qp + r}{p^e} \qquad (0 \leq r < p)$$

$$= \frac{q}{p^{e-1}} + \frac{r}{p^e}$$

induction

$$= c + \frac{r_1}{p^1} + \frac{r_2}{p^2} + \cdots + \frac{r_e}{p^e}.$$

That's it. Q.E.D.

To illustrate the algorithm we will compute

$$\int \frac{1}{(x+1)^2(x^2+1)}\,dx .$$

by partial fractions.

First $1 = \gcd\left((x+1)^2,\ x^2+1\right)$ in the ring of polynomials $\mathbb{R}[x]$. or $\mathbb{Q}[x]$.

Bézout: $\exists\ f(x), g(x) \in \mathbb{R}[x]$ where

$$1 = f(x)(x+1)^2 + g(x)(x^2+1)$$

how can we find these ?

We will use the "Euclidean Alg'thm"

Idea: Consider triples $f(x), g(x), h(x)$ where $f(x)(x+1)^2 + g(x)(x^2+1) = h(x)$.

Some easy examples

(1) $1(x+1)^2 + 0(x^2+1) = (x+1)^2$

(2) $0(x+1)^2 + 1(x^2+1) = x^2+1$

Step 1: Divide $(x+1)^2$ by x^2+1:

$$\begin{array}{r|l} & 1 \quad \text{quo} \\ \hline x^2+1 & x^2+2x+1 \\ & x^2+0+1 \\ \hline & 2x \quad \text{rem} \end{array}$$

$(x+1)^2 = 1 \cdot (x^2+1) + 2x$

Tells us to define new equation

(3) = 1 (1) − 1 (2)

(1) $1(x+1)^2 + 0(x^2+1) = (x+1)^2$

(2) $0(x+1)^2 + 1(x^2+1) = x^2+1$

(3) $1(x+1)^2 - 1(x^2+1) = 2x$

Next: Divide x^2+1 by $2x$:

$$
\begin{array}{r|l}
 & \tfrac{1}{2}x \quad \text{quo} \\
\hline
2x & x^2+1 \\
 & x^2+0 \\
\hline
 & \;\;1 \quad \text{rem.}
\end{array}
$$

$$(x^2+1) = \tfrac{1}{2}x\,(2x) + 1$$

$$1 = 1(x^2+1) - \tfrac{1}{2}x\,(2x)$$

Tells us to define new equation

(4) $= 1$ (2) $- \frac{1}{2}x$ (3)

(2) $0(x+1)^2 + 1(x^2+1) = x^2+1$

(3) $1(x+1)^2 - 1(x^2+1) = 2x$

(4) $-\frac{1}{2}x(x+1)^2 + (1+\frac{x}{2})(x^2+1) = 1$ ✓

Done.

Divide equation (4) by $(x+1)^2(x^2+1)$:

$$\frac{1}{(x+1)^2(x^2+1)} = \frac{-x/2\,\cancel{(x+1)^2}}{\cancel{(x+1)^2}(x^2+1)} + \frac{(1+x/2)\cancel{(x^2+1)}}{(x+1)^2\cancel{(x^2+1)}}$$

$$= \frac{-x/2}{x^2+1} + \frac{1+x/2}{(x+1)^2}$$

One more step: Divide $1+\frac{x}{2}$ by $x+1$. prime

$$\begin{array}{r|l} & \frac{1}{2} \quad \text{quo} \\ \hline x+1 & \frac{1}{2}\cancel{x} + 1 \\ & \cancel{\frac{1}{2}}x + \frac{1}{2} \\ \hline & \frac{1}{2} \quad \text{rem} \end{array}$$

$$= \frac{-x/2}{x^2+1} + \frac{\frac{1}{2}(x+1)+\frac{1}{2}}{(x+1)^2}$$

$$= \frac{-x/2}{x^2+1} + \frac{1/2}{(x+1)} + \frac{1/2}{(x+1)^2}$$ <u>Done.</u>

Finally, the integral is

$$-\frac{1}{2}\,\frac{1}{2}\ln(x^2+1) + \frac{1}{2}\ln|x+1| + \frac{1}{2}\left(\frac{-1}{x+1}\right).$$

Conclusion: If we can factor the denominator into degrees 1 & 2 then we can integrate any rational function.

Next Time:

So, <u>can</u> we factor the denominator?