HW4 due next Fri Apr 10.

Current Topic: F.T.A.

$\mathbb{R}$ version says every $f(x) \in \mathbb{R}[x]$ has prime factors of degrees 1 & 2.

History: Integrating rational functions. If $f(x), g(x) \in \mathbb{R}[x]$, and $g(x)$ has factors of deg 1 & 2 then by "partial fraction expansion" we can reduce the integral

$$\int \frac{f(x)}{g(x)}\,dx$$

to three basic forms:

- $\int \frac{1}{x+a}\,dx = \ln|x+a|$
- $\int \frac{1}{x^2+b^2}\,dx = \frac{1}{b}\arctan\left(\frac{x}{b}\right)$
- $\int \frac{x}{x^2+c^2}\,dx = \frac{1}{2}\ln\left(x^2+c^2\right).$

If FTA is true, then we can integrate *any* rational function in terms of arctan & ln.

$$\underbrace{(\text{trig}) \quad (\text{exp})}_{\text{"elementary functions"}}$$

Warning: $\int e^{-x^2}\,dx$ *cannot* be integrated in terms of elementary functions!

Today: Partial Fractions

DEF: A ring R is a "Euclidean domain" if

- R is an "integral domain"

  $ab = 0 \Rightarrow a = 0$ or $b = 0$.

- R has "division with remainder"

  $\forall a, b \in R,\ b \neq 0,\ \exists\, q, r \in R,$

$$\begin{cases} a = qb + r \\ r = 0 \text{ or } N(r) < N(b), \end{cases}$$

where $N : R \setminus 0 \to \mathbb{N}$ is some "size function"

[Remark: $R \setminus 0 = R \setminus \{0\}$
all nonzero elements of R.]

Examples: $\mathbb{Z}$ with $N(a) = |a|$.

$\mathbb{F}[x]$ with $N(f(x)) = \deg(f)$.

Examples of "Partial Fractions."

$$\frac{7}{15} = \frac{7}{3 \cdot 5} \overset{?}{=} \frac{?}{3} + \frac{?}{5}$$

can we un-add fractions?

Recall Bézout's Identity:
Given $\gcd(a,b) = d$, there exist

some (non-unique) $x, y \in \mathbb{Z}$ such that

$$d = ax + by.$$

If $\gcd(a,b) = 1$ then

$$1 = ax + by.$$

Divide both sides by $a \cdot b$:

$$\frac{1}{ab} = \frac{\cancel{a}x}{\cancel{a}\cdot b} + \frac{\cancel{b}y}{a\cdot \cancel{b}}$$

$$\frac{1}{ab} = \frac{y}{a} + \frac{x}{b}$$

For any $c \in \mathbb{Z}$:

$$\frac{c}{ab} = \frac{cy}{a} + \frac{cx}{b}.$$

Our Example: $\frac{7}{15}$.

$15 = 3 \cdot 5$

$\gcd(3,5) = 1 \implies 1 = 3x + 5y$

$1 = 3(-3) + 5(2)$

trial-and-error.

Divide both sides by 15:

$$\frac{1}{15} = \frac{\cancel{3}(-3)}{\cancel{15}_{5}} + \frac{\cancel{5}(2)}{\cancel{15}_{3}}$$

$$= \frac{2}{3} + \frac{-3}{5}.$$

Multiply both sides by 7:

$$\frac{7}{15} = \frac{14}{3} + \frac{-21}{5}.$$

This representation is not <u>unique</u>, but it becomes unique if we put the fractions in "proper from."

$$\frac{14}{3} = 4 + \frac{2}{3} \qquad 14 = (4)3 + 2$$

$$\frac{-21}{5} = -5 + \frac{4}{5} \qquad -21 = (-5)5 + 4$$

quotient is allowed to be < 0.

$$\frac{7}{15} = \left(4 + \frac{2}{3}\right) + \left(-5 + \frac{4}{5}\right)$$

$$= -1 + \frac{2}{3} + \frac{4}{5}$$

This representation is UNIQUE.
The unique partial fraction expansion of $7/15$.

One more issue: repeated prime factors.

$$12 = \underbrace{2 \cdot 2}_{\text{repeated.}} \cdot 3$$

$$\frac{7}{12} = \, ?$$

First: $12 = 4 \cdot 3$

$\gcd(4,3) = 1 \Rightarrow 1 = 4x + 3y$
$= 4(1) + 3(-1)$

Divide by 12:

$$\frac{1}{12} = \frac{4\cdot 1}{12} + \frac{3(-1)}{12} = \frac{1}{3} + \frac{-1}{4}$$

Multiply by 7:

$$\frac{7}{12} = \frac{7}{3} + \frac{-7}{4}$$ Now what?

Divide 7 by 3:

$$\frac{7}{3} = \frac{2\cdot 3}{3} + \frac{1}{3} = 2 + \frac{1}{3} \quad \checkmark$$

TRICK: $4 = 2^2$, 2 prime, so we should divide -7 not by 4, but by 2:

$$\frac{-7}{4} = \frac{(-4)2}{4} + \frac{1}{4} = -2 + \frac{1}{4}$$

OK, fine

(I chose the example badly).

More generally: $\frac{15}{4} = \frac{7\cdot 2 + 1}{4}$

$$= \frac{7}{2} + \frac{1}{4}$$

Observation: For p prime we should expect

$$\frac{a}{p^k} = ? + \frac{?}{p} + \frac{?}{p^2} + \cdots + \frac{?}{p^k}.$$

Theorem (Partial Fractions):

Let (R, N) be Euclidean domain.

Consider a fraction $\frac{a}{b}$, $a, b \in R$.

Denominator has unique prime fact'n.

$$b = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

Then $\exists$ some $c, r_{ij} \in R$ where

$$\begin{cases} \dfrac{a}{b} = c + \displaystyle\sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{r_{ij}}{p_i^j} \\ \forall i,j,\ r_{ij} = 0 \text{ or } N(r_{ij}) < N(p_i). \end{cases}$$

[Remark: Sometimes the numbers c, r_{ij} are <u>unique</u>, but we won't prove this]

Proof : Next Time.

Example for polynomials.

$$\frac{x^5 + x + 1}{(x+1)^2 (x^2+1)}$$

"prime" $x+1$ is repeated

We should expect expansion of form

$$f(x) + \frac{r_1(x)}{x^2+1} + \frac{r_2(x)}{(x+1)^2} + \frac{r_3(x)}{(x+1)^1}$$

where $\deg(r_1) < 2$

$\deg(r_2), \deg(r_3) < 1$

$r_1(x) = Ax + B$

$r_2(x) = C$

$r_3(x) = D$.