HW5 due before Friday's class.

See the more powerful hints.

Questions?

Problem 5: $\Phi_n(x) \in \mathbb{Z}[x] \ \forall n \geq 1$.

$\Phi_1(x) = x - 1$. ✓

$$\underbrace{x^n - 1}_{f,\ \mathbb{Z}} = \prod_{\substack{d \mid n \\ 1 \leq d \leq n}} \Phi_d(x) = \underbrace{\Phi_n(x)}_{q,\ \mathbb{C}} \underbrace{\prod_{\substack{d \mid n \\ 1 \leq d < n}} \Phi_d(x)}_{g,\ \text{assume } \mathbb{Z}}$$

Leading coeff. 1

Situation:

$f(x) = g(x)\, q(x)$

$f(x), g(x) \in \mathbb{Z}[x]$, $q(x) \in \mathbb{C}[x]$.

& g has leading coefficient 1.

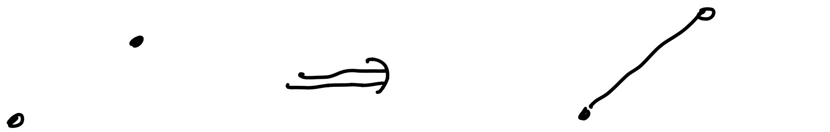
$$\xRightarrow[4(b)]{} \quad q(x) \in \mathbb{Z}[x].$$

New Topic: Impossible Constructions.

Euclid's Geometry is based on 4 principles of construction.

Start with 2 points. Then apply any of the following:

(1)



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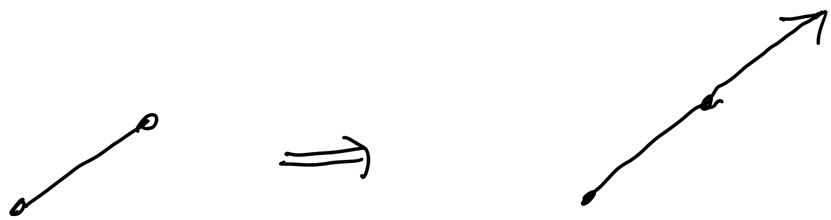

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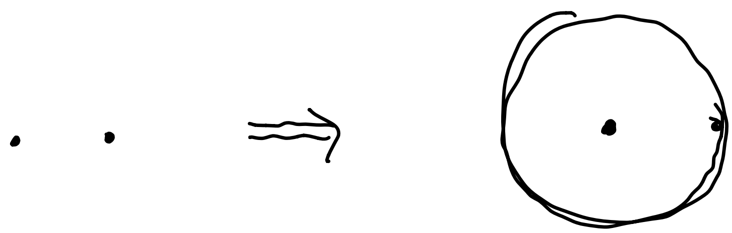

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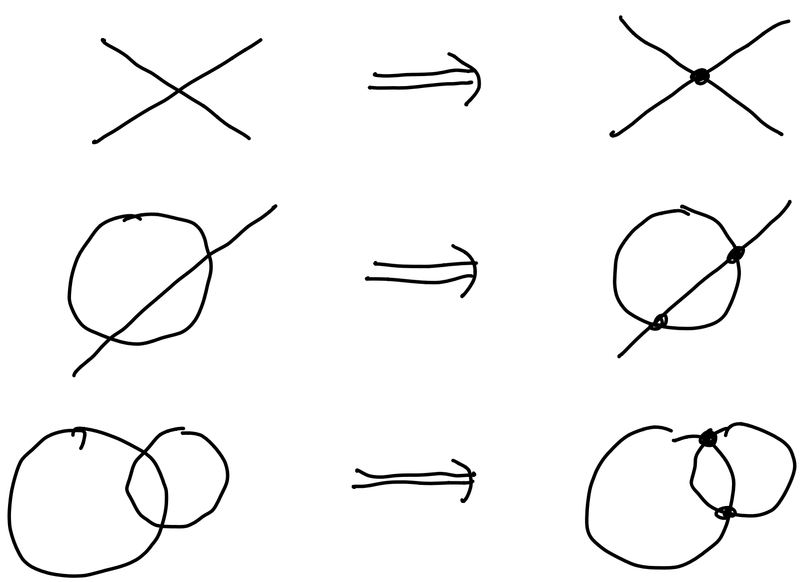
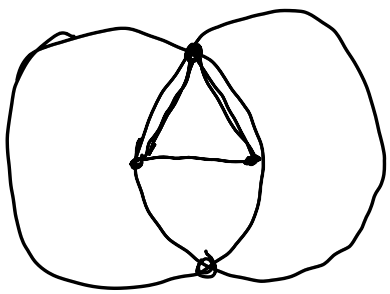
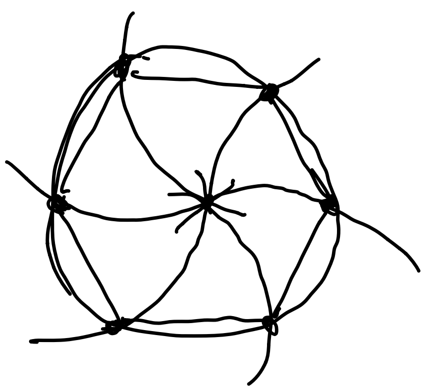


What kinds of geometric figures can be constructed using just ①, ②, ③, ④?

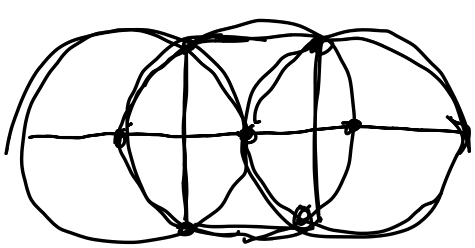
Example: Regular Polygons.



triangle ✓



regular hexagon.



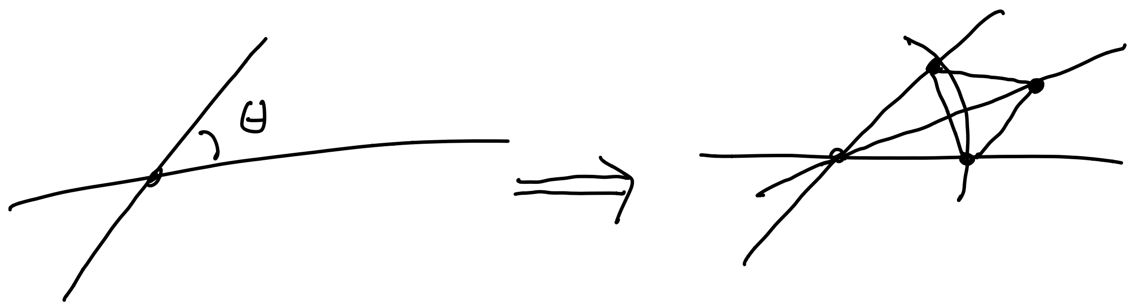
square (?)

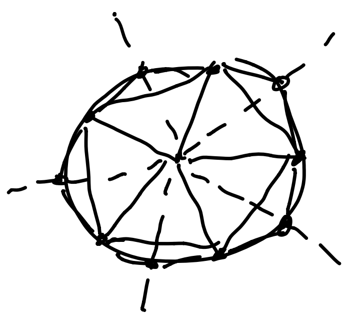
A pentagon is tricky but is also possible (Euclid).

General Rules:

- n-gon is cble $\Rightarrow$ $2^k n$-gon cble.

Proof: Bisect the angle.



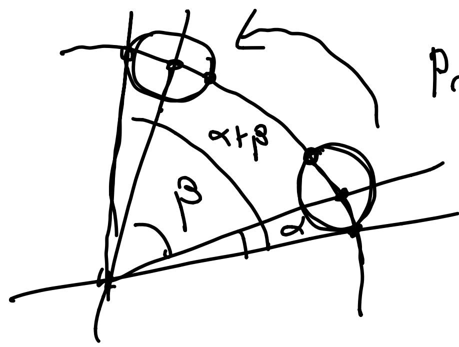


Double.

- Given $\gcd(m,n) = 1$,

$\left.\begin{array}{l} m\text{-gon} \\ n\text{-gon} \end{array}\right.$ cble $\Rightarrow$ mn-gon cble.

Proof: Add Angles.



Prop I.2

duplicate circle at new center.

m-gon
n-gon cble $\Rightarrow$ $\frac{2\pi}{n}$ & $\frac{2\pi}{m}$ cble

m, n coprime $\Rightarrow$ $mx + ny = 1$ $x, y \in \mathbb{Z}$.

Multiply both sides by $\frac{2\pi}{mn}$:

$$\underbrace{\frac{2\pi}{mn}}_{\text{cble}} = \underbrace{\left(\frac{2\pi}{n}\right)x + \left(\frac{2\pi}{m}\right)y}_{\text{cble}}.$$

cble $\Leftarrow$ cble

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We have shown that the following n-gons are constructible:

✓ 3, 4, 5, 6, 8, 10, 12, 15, 16, ...

What about

? 7, 9, 11, 13, 17, ...

The Greeks left this problem unsolved.

Turns out that 7-gon is impossible, but the proof requires totally new ideas.

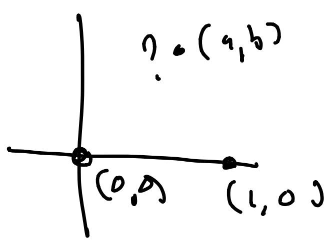
Descartes Changed the Rules:

Euclid : Geometry $\rightsquigarrow$ Algebra.

Descartes : Algebra $\rightsquigarrow$ Geometry.

Descartes translated problem of geometric constructibity into language of algebra.

Theorem: Suppose points $(0,0)$ & $(1,0)$ are given, we say that the point (a,b) is "constructible" if it can be obtained from $(0,0)$ & $(1,0)$ by repeated application of Euclid's constructions. ①, ②, ③, ④.



I claim that point (a,b) is cble $\Longleftrightarrow$ numbers a & b can be obtained from 0 & 1 using operations

$+, -, \times, \div, \sqrt{\ }$.

Let $\mathbb{Q}_{sqrt}$ be the set of such numbers.

e.g. $1+1+1 = 3 \in \mathbb{Q}_{sqrt}$

$\Rightarrow \sqrt{3} \in \mathbb{Q}_{sqrt}$

$\Rightarrow (5+\sqrt{3})/2 \in \mathbb{Q}_{sqrt}$

$\Rightarrow \left(1+\sqrt{(5+\sqrt{3})/2}\right)/6 \in \mathbb{Q}_{sqrt}$

$\vdots$

Observe $\mathbb{Q} \subseteq \mathbb{Q}_{sqrt}$ $(\subseteq \mathbb{C})$ is a <u>field</u> because it is closed under operations $+, -, \times, \div$.

<u>Proof</u> (Sketch): Two Directions.

- point (a,b) cble $\Rightarrow$ $a, b \in \mathbb{Q}_{sqrt}$

Proof: Every new point is obtained as intersection of lines & circles with Cartesian equations of the form

$ax + by = c$ $\qquad a, b, c \in \mathbb{Q}\text{sqrt}.$

$(x-a)^2 + (y-b)^2 = c^2$

The intersection of any two such shapes gives quadratic equations

$ax^2 + bx + c = 0$ & $dy^2 + ey + f = 0$

where $a, b, c, d, e, f \in \mathbb{Q}\text{sqrt}$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \in \mathbb{Q}\text{sqrt}$$

$$y = \frac{-e \pm \sqrt{e^2 - 4df}}{2d} \in \mathbb{Q}\text{sqrt}.$$

///

- $a, b \in \mathbb{Q}\text{sqrt} \Rightarrow (a,b)$ cble.

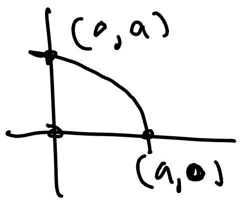
Enough to show:

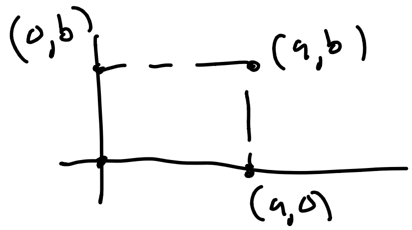
$a \in \mathbb{Q}\text{sqrt} \Rightarrow (a, 0)$ is cble.

[Because

$(a,0)$ cble $\Leftrightarrow$ $(0,a)$ cble.

(a,b) cble $\Longleftrightarrow$ $(a,0)$ & $(0,b)$ cble.]

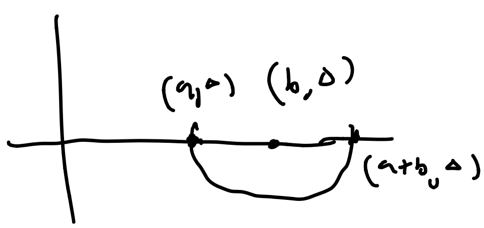


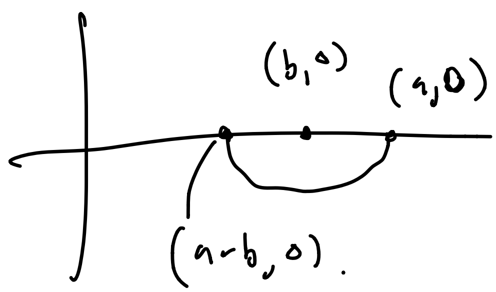


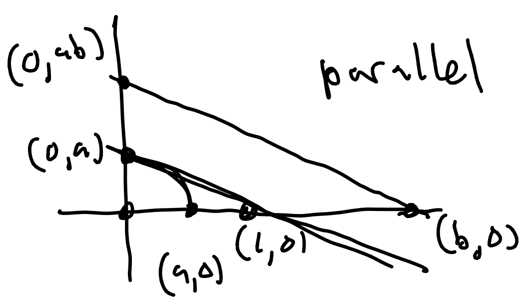
Euclid
I.12 & I.31

Given cble $(a,0)$ & $(b,0)$ we will construct $(a+b,0)$, $(a-b,0)$
$(ab,0)$, $(a/b,0)$ & $(\sqrt{a},0)$.

Descartes:



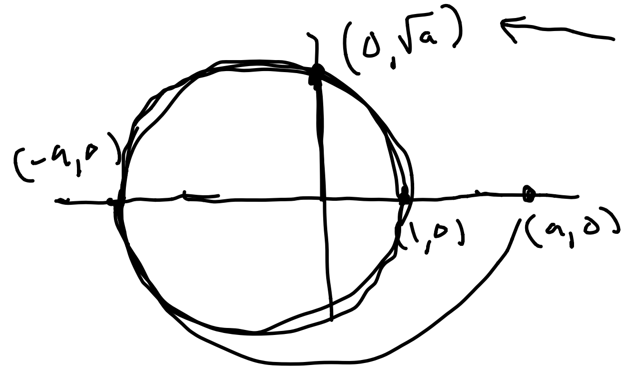




parallel Euclid I.31.

Division uses a similar diagram.

What about $\sqrt{\cdot}$?



can be proved using Pythagorean Theorem.

Q.E.D.

This translation of geometry into algebra allowed:

- Descartes to prove that a cube cannot be "doubled". (The "Delian problem")
- Gauss (-Wantzel) to prove that

regular n-gon cble $\iff \phi(n) = 2^k, \ k \in \mathbb{Z}$.

n	7	9	11	13	17
$\phi(n)$	6	6	10	12	16
cble?	N	N	N	N	Y

What?!

Through his study of the "cyclotomic equation" $x^n - 1 = 0$, Gauss accidentally discovered (at age 19) that $\cos\left(\frac{2\pi}{17}\right) \in \mathbb{Q}_{sqrt}$.

It follows that

$$\sin\left(\frac{2\pi}{17}\right) = \sqrt{1 - \cos^2\left(\frac{2\pi}{17}\right)} \in \mathbb{Q}_{sqrt}$$

and hence the regular 17-gon is constructible with straightedge & compass.

This was the subject of Gauss' very first publication.