Math 461 **Spring 2015**

Rings and Fields **Drew Armstrong**

I will avoid stating the formal definition of rings and fields in the lecture, but here it is in case you want to know.

Definition of a Ring. A ring is a set R together with two binary operations

$$+ : R \times R \to R \qquad \text{and} \qquad \cdot : R \times R \to R$$

and two special elements $0, 1 \in R$ $(0 \neq 1)$ that satisfy the following eight axioms:

(F1) $\forall\, a, b \in R, a + b = b + a$
(F2) $\forall\, a, b, c \in R, a + (b + c) = (a + b) + c$
(F3) $\forall\, a \in R, 0 + a = a$
(F4) $\forall\, a \in R, \exists\, b \in R, a + b = 0$
(F5) $\forall\, a, b \in R, ab = ba$
(F6) $\forall\, a, b, c \in R, a(bc) = (ab)c$
(F7) $\forall\, a \in R, 1a = a$
(F8) $\forall\, a, b, c \in R, a(b + c) = ab + ac$

Definition of a Field: We say that a ring R is a field if it satisfies one additional axiom:

(F9) $\forall\, 0 \neq a \in R, \exists\, b \in R, ab = 1$

Remarks:

- Some people omit axiom (F5) from the definition of a ring. Then the definition also includes *non-commutative* rings like the ring of square matrices. I find it more convenient to include (F5) in the definition and to explicitly remove it when necessary.
- Some people even omit axiom (F7) from the definition. I have less sympathy for that kind of thing.
- This definition is purely formal. An abstract ring doesn't need to have any kind of meaning or interpretation. In practice, however, we will always be talking about rings of integers ($\mathbb{Z}$) or polynomials ($\mathbb{F}[x]$ where $\mathbb{F}$ is $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$).
- The advantage of having a formal definition is that it forces us to see bigger patterns and ultimately it allows us to go further by compactifying large areas of mathematics into a tiny space.
- The axioms are just the beginning. From them one can develop a whole universe of "ring theory". To see what I mean, look at the table of contents of the book "Commutative Ring Theory" by Hideyuki Matsumura.

Exercise: Let R be a ring. Prove that for all $a \in R$ we have

$$0a = 0.$$