3/23/15

Today: Review
Wednesday: Exam 2.

Since Exam 1 we have been discussing applications of the Polar Form of complex numbers.

Recall de Moivre's Formula:

For all $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$ we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof: Rotating once by $n \cdot \theta$ is the same as rotating n times by θ. ///

More generally if we define

$$\operatorname{cis}(\alpha) := \cos(\alpha) + i\sin(\alpha)$$

then for all $\alpha, \beta \in \mathbb{R}$ we have

$$\operatorname{cis}(\alpha) \cdot \operatorname{cis}(\beta) = \operatorname{cis}(\alpha + \beta).$$

Proof: Rotating by β and then by α is the same as rotating once by $\alpha + \beta$. ///

This implies that

$$\text{cis}(t) = e^{kt}$$

for some constant k. To compute k we differentiate and evaluate at $t = 0$.

$$\text{cis}'(t) = k e^{kt}$$
$$\text{cis}'(0) = k e^{0} = k.$$

But $\text{cis}'(t) = -\sin t + i \cos t$

$$\text{cis}'(0) = -\sin 0 + i \cos 0 = i$$

We conclude that

$$e^{it} = \text{cis}(t) = \cos t + i \sin t.$$

This is called Euler's Formula. It is key to multiplying complex numbers.

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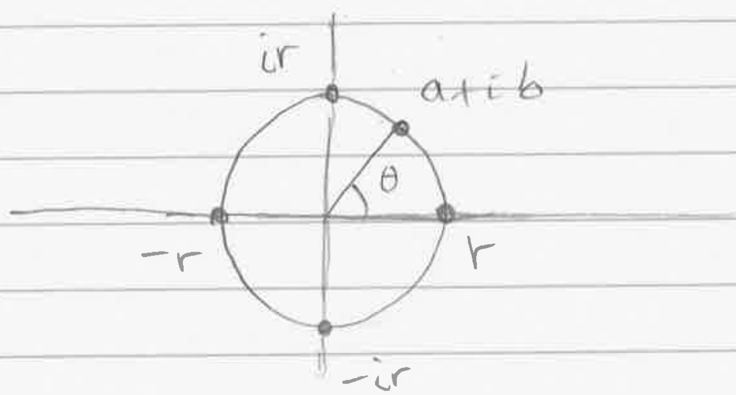
Polar Form:

Given $a+ib \in \mathbb{C}$, let

$$r := |a+ib| = \sqrt{a^2+b^2}$$

so that $a^2+b^2 = r^2$. This means that $a+ib$ lives on the circle of radius r centered at $0 \in \mathbb{C}$.



Let θ be the angle of $a+ib$ c.c.w. from the positive real axis. Then by definition of sine and cosine we have

$$a = r\cos\theta$$
$$b = r\sin\theta$$

We conclude that

$$
\begin{aligned}
a + ib &= r\cos\theta + ir\sin\theta \\
&= r(\cos\theta + i\sin\theta) \\
&= re^{i\theta}.
\end{aligned}
$$

This is called the polar form.

It is easy to multiply complex numbers in polar form. Given

$$u = r_1 e^{i\theta_1} \quad \& \quad v = r_2 e^{i\theta_2}$$

we have

$$
\begin{aligned}
uv &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\
&= r_1 r_2 e^{i\theta_1} e^{i\theta_2} \\
&= r_1 r_2 e^{i\theta_1 + i\theta_2} \\
&= r_1 r_2 e^{i(\theta_1 + \theta_2)}
\end{aligned}
$$

"multiply the lengths; add the angles"

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Roots of Unity :

To solve the equation $x^n = 1$ we write $x = re^{i\theta}$ in polar form. Then

$$(re^{i\theta})^n = 1$$
$$r^n e^{in\theta} = 1$$

$$\Longrightarrow \quad r = 1$$

$$\Longrightarrow \quad e^{in\theta} = 1$$

$$\Longrightarrow \quad n\theta = 2\pi k$$

$$\theta = 2\pi k/n \quad \text{for some } k \in \mathbb{Z}.$$

There are n solutions and we typically represent them by choosing

$$k = 0, 1, 2, \cdots, n-1.$$

$$\sqrt[n]{1} = \left\{ e^{2\pi i k/n} : k = 0, 1, 2, \cdots, n-1 \right\}$$

It is more efficient to define

$$\omega := e^{2\pi i/n}$$

so that $\omega^k = e^{2\pi i k/n}$ and the nth roots of 1 are

$$1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}.$$

Using Descartes' Factor Theorem gives

$$x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1})$$

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More generally we can compute the nth roots of any complex number $z \in \mathbb{C}$.

Fact: Let α be any particular nth root of z:

$$\alpha^n = z.$$

Then the full list of nth roots of z is

$$\alpha, \alpha\omega, \alpha\omega^2, \dots, \alpha\omega^{n-1}.$$

Proof: Exercise. ///

As a special case, let $z = b^n$ for some $b \in \mathbb{C}$. Then b is a particular nth root so the full list of nth roots is

$$\sqrt[n]{b^n} = \{ b, \omega b, \omega^2 b, \dots, \omega^{n-1} b \}$$

Using Descartes' Factor Theorem gives

$$x^n - b^n = (x-b)(x-\omega b)(x-\omega^2 b)\cdots(x-\omega^{n-1} b).$$

By evaluating at $x = a$ for any $a \in \mathbb{C}$ we obtain HW4 Problem 1.

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Now let $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$. We say that z is a primitive nth root of 1 if

- $z^n = 1$
- $z^m \neq 1$ for $m < n$.

Example: The primitive 4th roots of 1.

The 4th roots of 1 are

$$1,\ i,\ -1,\ -i.$$

1 is not primitive because $1^1 = 1$.
-1 is not primitive because $(-1)^2 = 1$.

The primitive 4th roots are

$$i \text{ and } -i.$$

Knowing the primitive roots allows us to factor $x^4 - 1$ over $\mathbb{Z}$.

$$x^4 - 1 = \underbrace{(x-1)}_{\substack{\text{primitive}\\ \text{1th roots}}} \underbrace{(x+1)}_{\substack{\text{primitive}\\ \text{2th roots}}} \underbrace{(x-i)(x+i)}_{\substack{\text{primitive}\\ \text{4th roots}}}$$

$$= (x-1)(x+1)(x^2+1)$$

$$= \Phi_1(x)\,\Phi_2(x)\,\Phi_4(x).$$

In general we have

$$\Phi_d(x) = \prod_{\substack{\zeta \text{ is a primitive} \\ d\text{th root of } 1}} (x - \zeta) \in \mathbb{Z}[x]$$

and

$$x^n - 1 = \prod_{d \text{ divides } n} \Phi_d(x)$$

Exercise: Use 5th roots of 1 to compute the exact values of

$$\cos\left(\frac{2\pi k}{5}\right) \text{ for all } k \in \mathbb{Z}.$$