1/12/15

Welcome to MTH 461!

Syllabus:

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- office: 533 Ungar
- office hours: TBA.

There is no textbook. I will scan all lecture notes and post them on my webpage:

www.math.miami.edu/~armstrong

There will be
- 6 HW assignments
- 3 in-class exams
- NO final exam

Your grade is based on

25% HW
25% Exam 1
25% Exam 2
25% Exam 3

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Course Topic:

The official title of MTH 461 is

"Survey of Modern Algebra"

I don't like this title. I think a better title is

"Pre-Abstract Algebra"

This title is intended to be provocative.
[Mention "Calculus" vs. "Pre-Calculus"]

It is not well-known outside of mathematics that the definition of "algebra" was changed around 1930.

- Before 1930, "algebra" is the study of solving equations.

- After 1930, (von der Warden's textbook "Moderne Algebra"), "algebra" is the study of abstract structure in mathematics.

Examples of abstract structures are:

groups, rings, fields, vector spaces, modules, categories, etc.

These all have formal definitions that will make no sense the first time you see them.

There are a few possible ways to organize a math course:

1. Abstract Structure.

2. Concrete Examples / Applications

3. History and Motivation.

The traditional teaching style is

$$① \to ② .$$

In MTH 461, I will use

$$③ \to ② \to ① .$$

In this spirit, the topic of 461 will be:

"The history of solving equations, with a view towards abstract algebra."

BEGIN.

We want solve equations. What kind of equations?

Example: Solve $2x+3=0$ for x.

$$\begin{aligned} 2x+3 &= 0 \\ 2x &= -3 \\ x &= -3/2. \end{aligned}$$

Example: Solve $ax+b=0$ for x.

- If $a \neq 0$ then we have

$$\begin{aligned} ax+b &= 0 \\ ax &= -b \\ x &= -b/a, \end{aligned}$$

- If $a = 0$ then we have

$$0x + b = 0.$$

  - If $b \neq 0$ then there is <u>no</u> solution
  - If $b = 0$ then <u>every</u> possible value of x is a solution.

<u>Example</u>: Solve the simultaneous equations

$$\begin{aligned} &(i) \quad 2x + 3y + 1 = 0 \\ &(ii) \quad x + y - 2 = 0. \end{aligned}$$

From (ii) we have $x = 2 - y$.
Substitute into (i) to get

$$\begin{aligned} 2(2-y) + 3y + 1 &= 0 \\ 4 - 2y + 3y + 1 &= 0 \\ y + 5 &= 0 \\ y &= -5. \end{aligned}$$

Substitute into (ii) to get

$$\begin{aligned} x + (-5) - 2 &= 0 \\ x - 7 &= 0 \\ x &= 7. \end{aligned}$$

Remark: A "linear equation" has the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

where
- $a_1, a_2, \ldots, a_n, b$ are constants
- $x_1, x_2, \ldots, x_n$ are unknowns.

The study of solving systems of linear equations is called <u>linear algebra</u> (MTH 210), as you know.

We (humans) are quite good at linear algebra.

In MTH 461 we will consider "non-linear equations".

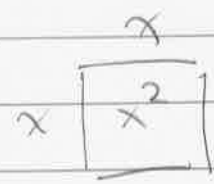
Example from "Hisab al-jabr w'al-muqabala" (Book of Calculation by Completion and Reduction), by al-Khwarizmi (AD 820):

Solve the equation

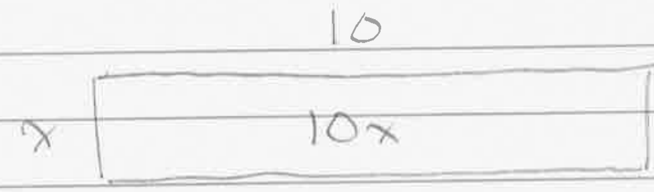
$$x^2 + 10x = 39 \quad \text{for } x.$$

Al-Khwarizmi used a geometric argument to solve this.

$x^2 =$



$10x =$



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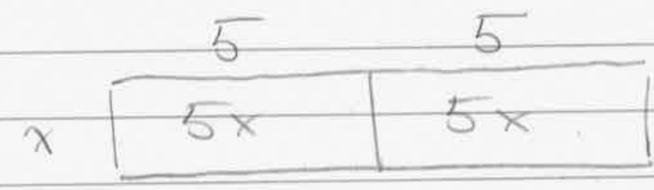
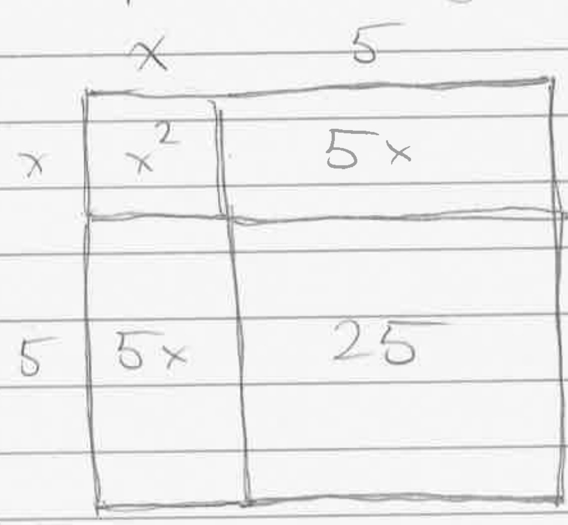


Now "complete the square"



The area of the big square is

$$(x+5)^2 = x^2 + 5x + 5x + 25$$

$$= \underbrace{x^2 + 1\,x} + 25$$

$$= 39 + 25$$

$$= 64$$

hence its side length is

$$x + 5 = 8.$$

We conclude that $x = 3$.

Remark: Al-Khwarizmi did not use negative numbers, so for him there were three <u>different</u> kinds of quadratic equations:

$$a + x^2 = bx$$
$$a = x^2 + bx$$
$$x^2 = a + bx.$$

Each required a different method of solution.

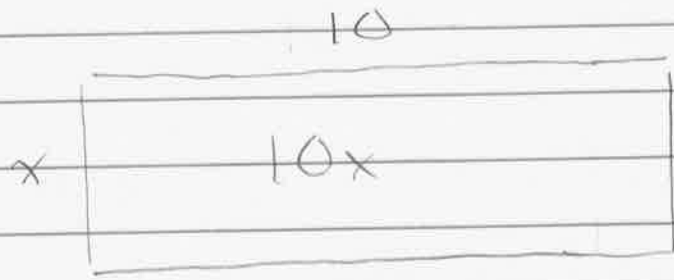
Thinking Problem:

Give a geometric solution to

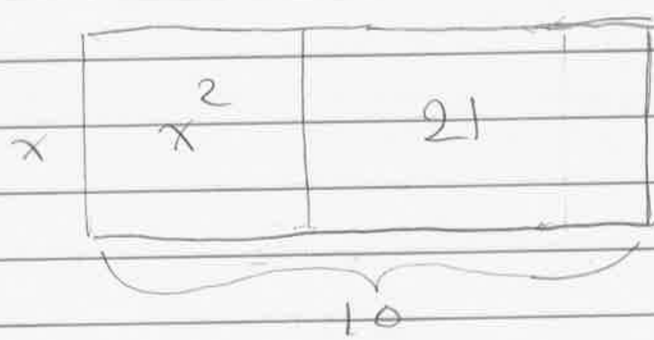
$$x^2 + 21 = 10x$$

without using negative numbers.

[Hint: Consider a rectangle of area $10x$ $(= x^2 + 21)$



and cut a square off the side



NOW WHAT?]

1/14/15

MTH 461

"Pre-Abstract Algebra"

Office Hours and HW 1 still TBA

Last time I mentioned al-Khwarizmi (AD 780 – 850) and his book

"... ḥisāb al-jabr w'al-muqābala"

(calculation by completion and reduction)

written around AD 825 in Baghdad.

Remarks:

- "al-jabr" = "algebra"
- "al-Khwarizmi" = "algorithm"
- al-jabr (completion) means <u>adding</u> a term to both sides of an equation to remove negative quantities:

$$2x + 5 = 8 - 3x \implies 5x + 5 = 8$$

- al-muqabala (reduction) means <u>subtracting</u> a positive term from both sides to remove it from one side:

$5x + 5 = 8 \implies 5x = 3$

- Al-Khwarizmi had no symbolic notation, so instead of $x^2 + 10x = 39$ he would write:

"one square [māl] and 10 roots of the same equals 39 direhems [a unit of currency]".

- Al-Khwarizmi did not use negative numbers, so for him there were three different kinds of quadratic equations:

(1) $x^2 + bx = a$
(2) $x^2 + a = bx$
(3) $a + bx = x^2$

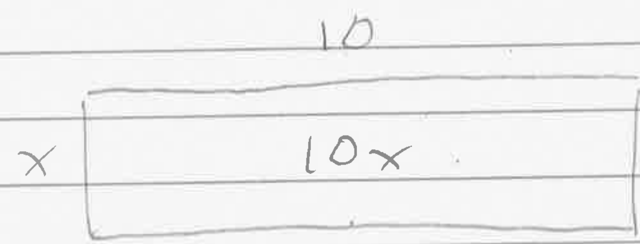
Each required a different method of solution.

Last time we saw his example of type (1):

Solve $x^2 + 10x = 39$ for x.

$x^2 =$

$10x =$



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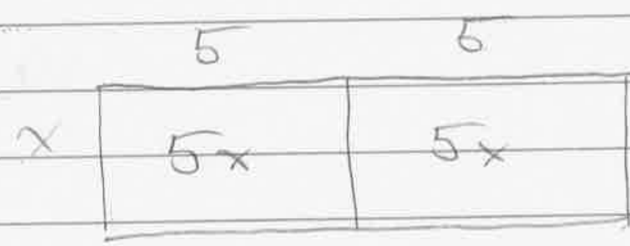
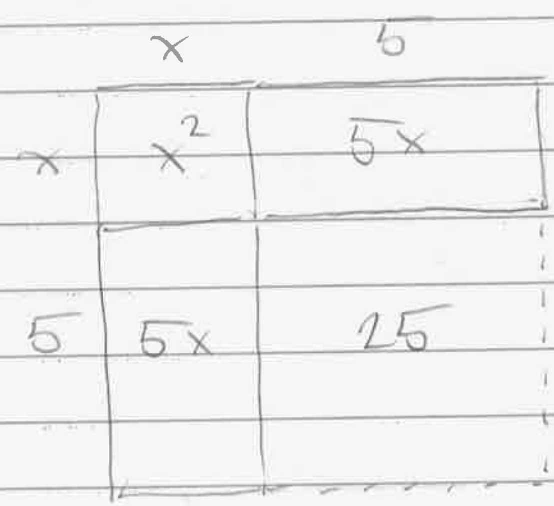


(Trick)

"Complete the square"



Area of big square is

$$\begin{aligned}(x+5)^2 &= x^2+5x+5x+25 \\ &= \underbrace{x^2+10x}\ +25 \\ &= 39 \quad +25 \\ &= 64\end{aligned}$$

so side length is

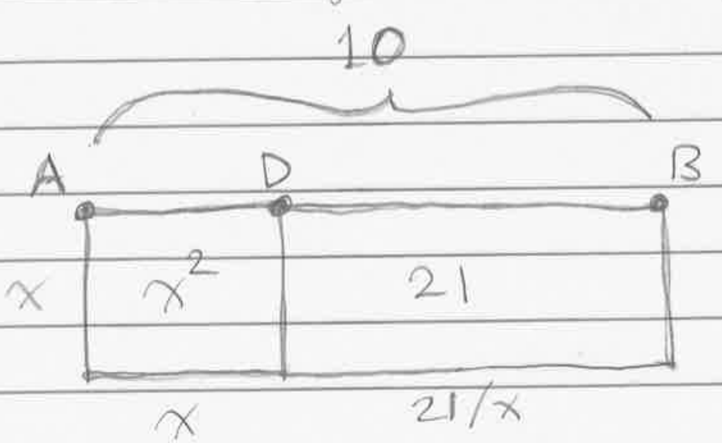
$$x+5 = 8$$

We conclude that $x = 3$.

Type ② is quite a bit trickier.

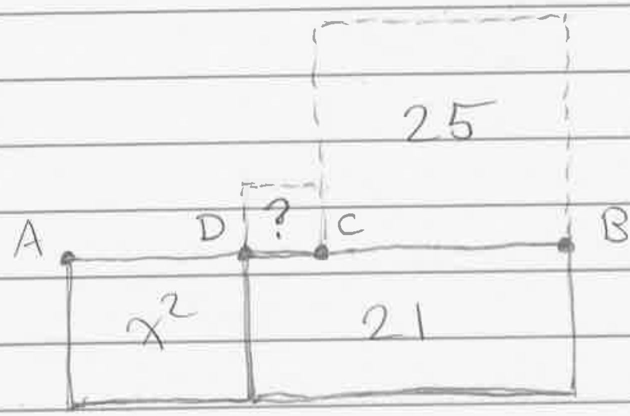
Example: Solve $x^2+21 = 10x$ for x.

We draw rectangle of area $10x$ $(=x^2+21)$.



Now let C be the midpoint of AB.
There are two cases.

Case 1: C is to the right of D.



Al-Khwarizmi gave a geometric argument that area ? + area 21 = area 25.

Since area ? $= (5-x)^2$, we have

$$
\begin{aligned}
(5-x)^2 + 21 &= 25 && \text{al-muqabala} \\
(5-x)^2 &= 4 \\
5-x &= 2 \\
5 &= 2 + x && \text{al-jabr} \\
3 &= x && \text{al-muqabala}
\end{aligned}
$$

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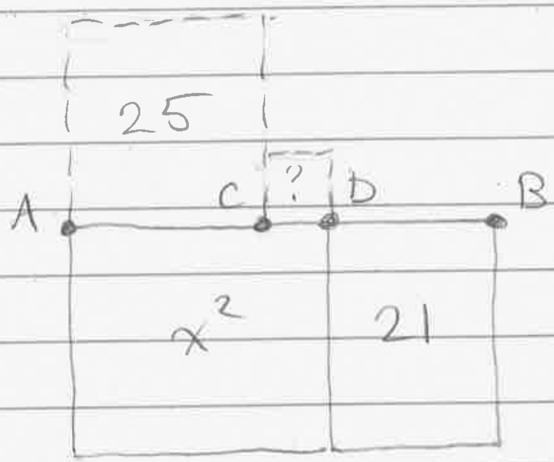
Case 2: C is to the left of D.



Al-Khwarizmi gave a (different) geometric argument that

$$\text{area } ? + \text{area } 21 = \text{area } 25 .$$

Since area ? $= (x-5)^2$ we have

$$(x-5)^2 + 21 = 25 \qquad \text{al-muqabala}$$
$$(x-5)^2 = 4$$
$$x - 5 = 2$$
$$x = 7 \qquad \text{al-jabr.}$$

So the equation $x^2 + 21 = 10x$ has <u>two</u> solutions

$$x = 3 \text{ and } x = 7 .$$

[Thinking Problem : Fill in the geometric arguments.]

We won't do type (3) :)

Summary: In AD 825, this kind of thing was not easy.

With symbolic notation and negative numbers it becomes much easier.

<u>Theorem</u>: The general quadratic eq$^{\underline{n}}$

$$ax^2 + bx + c = 0$$

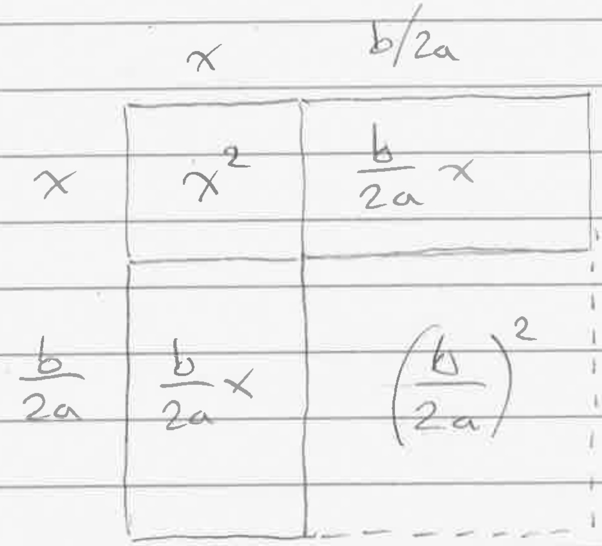
has solution $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

Proof: If $a = 0$ then this is just a linear equation, so we assume $a \neq 0$.

Divide by a to get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Now we "complete the square":



(This is just an analogy. We don't care if $(b/2a)x$ is negative.)

Area of big square is

$$x^2 + \frac{b}{2a}x + \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2$$

$$= \underbrace{x^2 + \frac{b}{a}x} + \frac{b^2}{4a^2}$$

$$= \quad -\frac{c}{a} \quad + \frac{b^2}{4a^2}$$

$$= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$$

So the side length is

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= \frac{1}{2a}\sqrt{b^2 - 4ac}$$

We conclude that

$$x = -\frac{b}{2a} + \frac{1}{2a}\sqrt{b^2 - 4ac}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$\blacksquare$

Thinking Problem: This looks like just one solution, but we know that some quadratic equations have two solutions.

What's going on?

1/16/15

HW 1 and Office Hours: still TBA.

Last time we used a geometric analogy to solve the general qudratic equation:

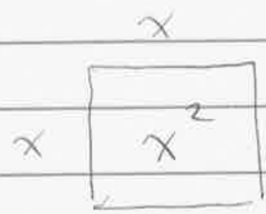
$$ax^2 + bx + c = 0.$$

We assume that $a \neq 0$ and divide by a to get

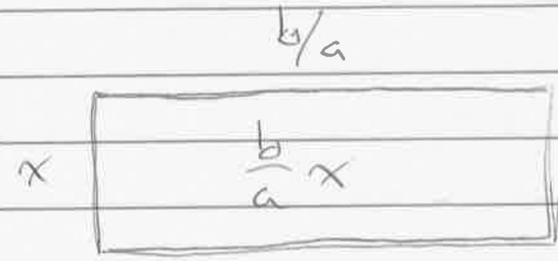
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

we think of:

$x^2 =$



$\frac{b}{a}x =$



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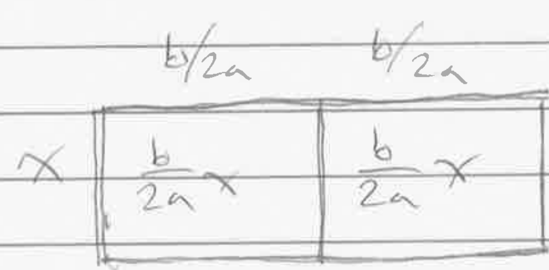
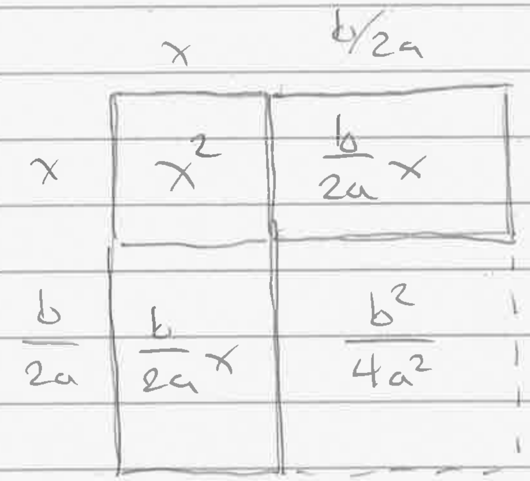


Then we "complete the square":



Area of the big square is

$$\left(x+\frac{b}{2a}\right)^2 = x^2+\frac{b}{2a}x+\frac{b}{2a}x+\frac{b^2}{4a^2}$$

$$= \underbrace{x^2+\frac{b}{a}x}+\frac{b^2}{4a^2}$$

$$= \frac{-c}{a}+\frac{b^2}{4a^2}$$

$$= \frac{b^2-4ac}{4a^2}.$$

Take "the" square root to get

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{1}{2a}\sqrt{b^2 - 4ac}$$

$$x = -\frac{b}{2a} + \frac{1}{2a}\sqrt{b^2 - 4ac}$$

$$\boxed{x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}}$$

Now we have to think:

- Does this agree with previous knowledge?

Example: Solve $x^2 + 10x = 39$.

$$x^2 + 10x - 39 = 0$$

$$\Longrightarrow \quad x = \frac{-10 + \sqrt{256}}{2}$$

$$= \frac{-10 + 16}{2} = \frac{6}{2} = 3 \quad \checkmark$$

Example: Solve $x^2 + 21 = 10x$.

$$x^2 - 10x + 21 = 0$$

$$\Longrightarrow \quad x = \frac{+10 + \sqrt{16}}{2}$$

$$= \frac{10+4}{2} = \frac{14}{2} = 7 \ \checkmark$$

OK, but we know that $x = 3$ is also a solution. Where is it?

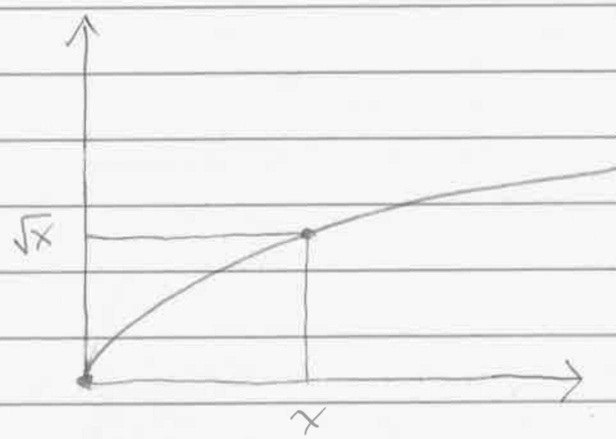
Problem: There is no such thing as <u>the</u> square root of a number.

In Calculus you may have seen

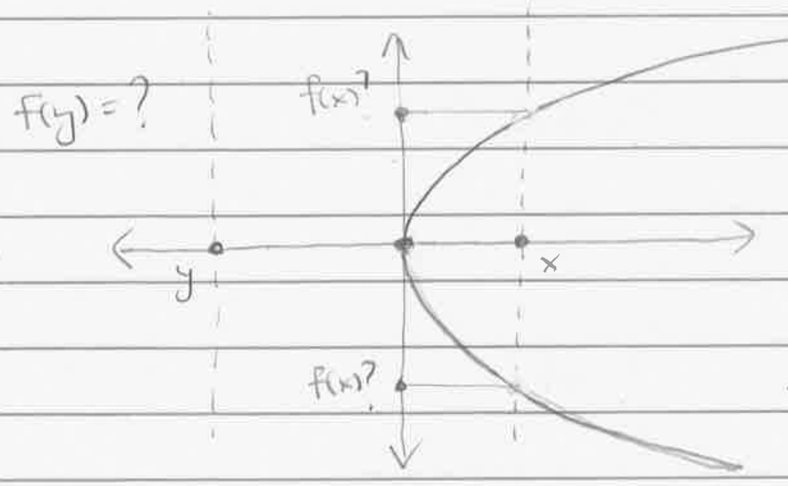
$$f(x) = \sqrt{x}$$

called a <u>function</u> but it is not. We can make it into a function by restricting the domain and range to non-negative real numbers $\mathbb{R}_{\geq 0}$.

$$f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$
$$x \longmapsto \sqrt{x}$$



Thus f was a function for al-Khwarizmi, because he did not use negative numbers. But it is not a function for us!



- when $x < 0$, $\sqrt{x}$ does not exist.
- when $x > 0$, $\sqrt{x}$ has two different values.
- The only number with a unique square root is $\sqrt{0} = 0$.

If we take this into account, then the "formula"

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

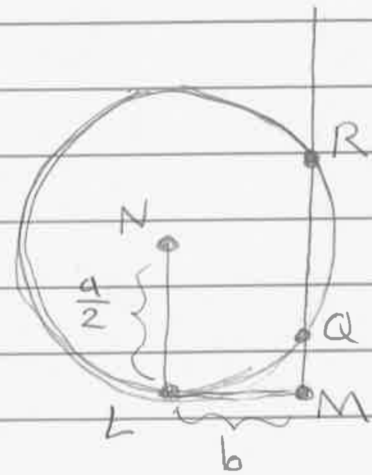
represents either 0, 1 or 2 values, depending on whether $b^2 - 4ac$ is < 0, $= 0$, or > 0.

[Jargon : $b^2 - 4ac$ is called the discriminant of the equation.]

René Descartes (1596–1650) gave a geometric explanation of this phenomenon.

From "La Géometrie" (1637):

Intersect a line and circle.



He showed that the lengths $\overline{MQ}$ and $\overline{MR}$ are solutions to the quadratic equation

$$x^2 - ax + b^2 = 0.$$

The Quadratic formula says

$$x = \frac{a + \sqrt{a^2 - 4b^2}}{2}$$

The discriminant is $a^2 - 4b^2$

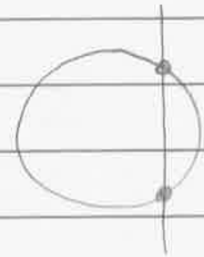
There are 2 solutions when

$a^2 - 4b^2 > 0$

$a^2 > 4b^2$

$a > 2b$

$a/2 > b$



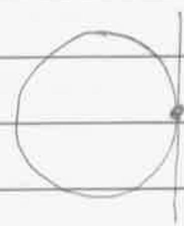
There is a unique solution when

$a^2 - 4b^2 = 0$

$a^2 = 4b^2$

$a = 2b$

$a/2 = b$



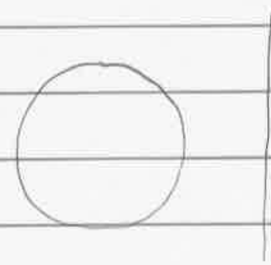
There is no solution when

$a^2 - 4b^2 < 0$

$a^2 < 4b^2$

$a < 2b$

$a/2 < b$



because in this case the circle and line do not intersect.

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1/21/15

HW 1 due Fri Jan 30 in class.

My Office Hours (Ungar 533)

- Mon 1-2pm
- Wed 2-3pm
- and by appointment.

Last week we developed

☆ The Quadratic Formula:

Consider the equation

$$ax^2 + bx + c = 0.$$

If $a \neq 0$, then the solution is given by the formula

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

This formula might represent 0, 1, or 2 real numbers, depending if the discriminant $b^2 - 4ac$ is <0, $=0$, or >0.

Next we will consider the general cubic equation

$$ax^3 + bx^2 + cx + d = 0.$$

Questions:

- Is there a formula for the solution?
- Is there a "discriminant" that tells us qualitative properties of the solution?

Quadratic equations were solved in ancient times and the general solution was obtained by Islamic mathematicians (including al-Khwarizmi) in the middle ages. They failed to solve cubic equations.

The "cubic formula" was discovered in the 1500s in Italy, and this marked the beginning of "modern mathematics" (the first time modern Europeans solved a problem that the ancients were unable to solve).

Before plunging into the general solution, let's look at some easier cases.

Example: Solve the equation

$$x^3 - 2x^2 - 5x + 6 = 0 \quad \text{for } x.$$

What can we do?
Try to *guess* a solution.

Note that $x = 1$ works:

$$1^3 - 2 \cdot 1^2 - 5 \cdot 1 + 6 = 0 \quad \checkmark$$

Now what?

[René Descartes (1596-1650) was the first person to use our modern notation for polynomials

$a, b, c, \cdots$	$x, y, z, \cdots$
constants	variables.

He observed a fundamental algebraic property:

Let $f(x)$ be a polynomial. If $f(r) = 0$, then the polynomial $f(x)$ is "divisible" by $(x-r)$:

$$f(x) = (x-r)\, g(x)$$

[where $g(x)$ is another polynomial (the "quotient").]

This means that our polynomial

$$f(x) = x^3 - 2x^2 - 5x + 6$$

is divisible by $(x-1)$. Let's compute the quotient.

[Do you remember how to perform long division of polynomials?]

$$
\begin{array}{r|l}
 & x^2 - x - 6 \\
\hline
x - 1 & \cancel{x^3} - 2x^2 - 5x + 6 \\
 & \underline{\cancel{x^3} - x^2 + 0 + 0} \\
 & \quad -\cancel{x^2} - 5x + 6 \\
 & \quad \underline{-\cancel{x^2} + x + 0} \\
 & \qquad\quad -6x + 6 \\
 & \qquad\quad \underline{-6x + 6} \\
 & \qquad\qquad\quad 0
\end{array}
$$

The quotient is $x^2 - x - 6$; the remainder is 0 (as expected).

$$x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 - x - 6) + 0.$$

So what? This simplifies the equation we're trying to solve:

$$(x-1)(x^2 - x - 6) = 0.$$

This means that <u>either</u> $x - 1 = 0$, <u>or</u>

$$x^2 - x - 6 = 0,$$

and this we can solve with the QF.

$$x = \frac{+1 + \sqrt{25}}{2}$$

$$= \frac{1+5}{2} \quad \text{OR} \quad \frac{1-5}{2}$$

$$= 3 \quad \text{OR} \quad -2 .$$

Use Descartes' trick again to write our equation as

$$(x-1)(x+2)(x-3) = 0 .$$

The solutions are

$$x = 1, -2, 3$$

and there can be no other solution.

[Why not?]

This leads to a general method.

☆ Method to Solve Cubic Equations:

Let $f(x)$ be a cubic polynomial. To solve $f(x) = 0$ we

- Guess a solution $f(r) = 0$.
- Use Descartes' trick to write $f(x) = (x-r)\,g(x)$, where $g(x)$ is a quadratic polynomial.
- Use the Quadratic Formula to solve $g(x) = 0$.

There are 2 issues:

1. What if we can't guess a solution?

2. Why does Descartes' trick work?

I prefer to deal with 2 first. This will be our first theorem of the course.

☆ Descartes' Factor Theorem (1637):

Let $f(x)$ be a polynomial with real coefficients and let α be a real number. Then we have

$$f(\alpha) = 0 \iff f(x) \text{ is divisible by } (x-\alpha).$$

Proof: The direction $\Leftarrow$ is easy.

Suppose that $f(x)$ is divisible by $(x-\alpha)$, i.e., we have $f(x) = (x-\alpha)\,g(x)$, where $g(x)$ is another polynomial with real coefficients. Then we have

$$\begin{aligned} f(\alpha) &= (\alpha-\alpha)\,g(\alpha) \\ &= 0 \cdot g(\alpha) \\ &= 0 \end{aligned}$$

///

The direction $\Rightarrow$ is harder.

To be continued ...

1/23/15

HW 1 due Fri Jan 30

Office Hours: Mon 1-2pm, Wed 2-3pm.

Right now we are trying to solve cubic equations:

$$ax^3 + bx^2 + cx + d = 0.$$

Last time we discussed a method.

Let $f(x)$ be a cubic polynomial. To solve the equation $f(x) = 0$, we

- Guess a solution $f(r) = 0$.
- Divide $f(x)$ by $(x-r)$ using long division to obtain $f(x) = (x-r)g(x)$, where $g(x)$ is a quadratic polynomial
- Solve the equation $g(x) = 0$ using the Quadratic Formula. If the solution is $x = s$ or t, then the solution of $f(x) = 0$ is

$$x = r \text{ or } s \text{ or } t.$$

///

There are 2 issues to discuss:

(1) What if we can't guess a solution?

(2) If $f(r)=0$, why does it follow that $f(x)=(x-r)g(x)$ for some $g(x)$?

We will deal with (2) first.

Let $\mathbb{R}$ be the set of real numbers (numbers with decimal expansions).

In modern terms we say that $\mathbb{R}$ is a field because real numbers can be added, subtracted, multiplied, and divided, subject to all of the familiar rules, e.g.,

$$a(b+c) = ab + ac.$$

A polynomial over $\mathbb{R}$ is any expression of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where $a_0, a_1, \ldots, a_n$ are real numbers (called the coefficients). If $a_n \neq 0$ we say the polynomial has degree n.

Q: What is x?

A: It is just a placeholder. We may interpret it as a real number at some point, or we may not. We may interpret it as "some other kind of number".

It's mysterious.

Polynomials can be added, subtracted, and multiplied:

Consider two polynomials

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

We assume that only finitely many coefficients are $\neq 0$, otherwise we would call them power series.

We define addition and subtraction:

$$f(x)+g(x) := (a_0+b_0)+(a_1+b_1)x + \cdots$$

$$f(x)-g(x) := (a_0-b_0)+(a_1-b_1)x + \cdots$$

Multiplication is trickier:

$$f(x)g(x) := \sum_{k\geq 0} \left(\quad ? \quad \right) x^k$$

$$= \sum_{k\geq 0} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k$$

Explicitly, we have

$$(a_0+a_1x+a_2x^2+\cdots)(b_0+b_1x+b_2x^2+\cdots)$$

$$= (a_0b_0) + (a_0b_1+a_1b_0)x$$

$$+ (a_0b_2+a_1b_1+a_2b_0)x^2$$

$$+ (a_0b_3+a_1b_2+a_2b_1+a_3b_0)x^3$$

$$+ \cdots$$

//

Let $\mathbb{R}[x]$ denote the set of polynomials over the field $\mathbb{R}$. In modern terms we say that $\mathbb{R}[x]$ is a <u>ring</u> because its elements can be added, subtracted, and multiplied, subject to the usual rules.

But $\mathbb{R}[x]$ is <u>not</u> a field because we can't divide.

Example: 1 and $1-x$ are polynomials, <u>but</u>

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

is <u>not</u> a polynomial (it's a power series).

Do you know another example of a ring?

Yes you do.

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

"The ring of integers".

Integers can be added, subtracted, and multiplied, but not divided.

Example: 1 and 2 are integers, but

$$\frac{1}{2}$$

is not an integer.

Deep Fact: The rings $\mathbb{Z}$ & $\mathbb{R}[x]$ are structurally very similar. This can be expressed by a pair of theorems.

☆ "Division Theorem" for $\mathbb{Z}$:

Consider $a, b \in \mathbb{Z}$ with $b \neq 0$. Then there exist unique $q, r \in \mathbb{Z}$ such that

- $a = qb + r$
- $|r| < |b|$.

q is for quotient, r is for remainder.

☆ "Division Theorem" for $\mathbb{R}[x]$:

Given $f(x), g(x) \in \mathbb{R}[x]$ with $g(x) \neq 0$, there exist <u>unique</u> $q(x), r(x) \in \mathbb{R}[x]$ such that

- $f(x) = q(x)g(x) + r(x)$
- $r(x) = 0$ or $\text{degree}(r) < \text{degree}(g)$.

These theorems are not very easy to prove, but they are very easy to believe because we can <u>compute</u> the quotient and remainder using

"Long division".

Example: Divide $x^3 + x + 1$ by $x - 2$.

$$
\begin{array}{rl}
 & \quad x^2 + 2x + 5 \\ \cline{2-2}
x - 2 & \big) \cancel{x^3} + 0x^2 + x + 1 \\
 & \quad \underline{\cancel{x^3} - 2x^2} \\
 & \qquad \cancel{2x^2} + x + 1 \\
 & \qquad \underline{\cancel{2x^2} - 4x + 0} \\
 & \qquad\qquad \cancel{5x} + 1 \\
 & \qquad\qquad \underline{\cancel{5x} - 10} \\
 & \qquad\qquad\quad 11
\end{array}
$$

DONE.

We conclude that

$$(x^3 + x + 1) = \underbrace{(x^2 + 2x + 5)}_{\text{quotient}}(x - 2) + \underbrace{(11)}_{\text{remainder}}$$

Note that

$$\deg(11) < \deg(x-2)$$

$$0 < 1$$

as expected!

We say that $f(x)$ is divisible by $g(x)$ if the remainder of $f(x)$ divided by $g(x)$ is zero.

Q: What does it mean that $x^3 + x + 1$ is not divisible by $x - 2$?

1/26/15

HW 1 due Friday.

Office Hours: Mon 1-2, Wed 2-3.

Math Club Today, 6:30pm, Ungar 402.

We are taking a digression through the theory of rings and fields, so that we can state and prove Descartes' Factor Theorem.

Recall: A ring is a set R with two binary operations

$$+ : R \times R \to R$$
$$\cdot : R \times R \to R$$

and two special elements $0, 1 \in R$ that satisfy eight axioms

[See the Handout.]

We say that a ring R is a field if it satisfies one further axiom:

$\downarrow$

- For all nonzero $a \in R$, there exists an element $b \in R$ such that

$$ab = 1.$$

I will often use the letter $\mathbb{F}$ to refer to a general field.

Warning: The theory of rings is extremely rich. In this course we are interested in two very special rings

$$\mathbb{Z} \quad \text{and} \quad \mathbb{F}[x],$$

where $\mathbb{F}$ is a field, probably $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.

These two rings share a very important property called

"Division With Remainder"

Example: Divide the polynomial $x^3 - x^2 - x - 1$ by the polynomial $x - 2$.

$$\begin{array}{r|l}
 & x^2 + x + 1 \\ \hline
x - 2 & \cancel{x^3} - x^2 - x - 1 \\
 & \cancel{x^3} - 2x^2 + 0 + 0 \\ \hline
 & \cancel{x^2} - x - 1 \\
 & \cancel{x^2} - 2x + 0 \\ \hline
 & \cancel{x} - 1 \\
 & \cancel{x} - 2 \\ \hline
 & 1
\end{array}$$

We conclude that

$$(x^3 - x^2 - x - 1) = \underbrace{(x^2 + x + 1)}_{\text{quotient}}(x-2) + \underbrace{1}_{\text{remainder}}.$$

Q: What does it mean that the remainder is not zero?

A: Evaluate both sides at $x = 2$:

$$(2)^3 - (2)^2 - (2) - 1 = (7)(\overset{0}{\cancel{2-2}}) + 1$$

$$(2)^3 - (2)^2 - (2) - 1 = 1.$$

It means that 2 is not a root of

$$x^3 - x^2 - x - 1.$$

This leads to the first theorem of algebra.

☆ Descartes' Factor Theorem (1637):

Let $\mathbb{F}$ be a field. Consider a polynomial $f(x) \in \mathbb{F}[x]$ and a constant $\alpha \in \mathbb{F}$. Then we have

$$f(\alpha) = 0 \iff f(x) \text{ is divisible by } x - \alpha.$$

Proof: Use the division theorem to divide $f(x)$ by $(x-\alpha)$. We obtain

$$f(x) = q(x)(x-\alpha) + r(x)$$

where $r(x) = 0$ or $\deg(r) < \deg(x-\alpha)$, i.e. $\deg(r) = 0$. So $r(x)$ is just a constant $r \in \mathbb{F}$.

$$(*) \quad f(x) = q(x)(x-\alpha) + r.$$

By definition we say that $(x-\alpha)$ divides $f(x)$ if and only if $r=0$. Now evaluate (*) at $x=\alpha$ to get

$$f(\alpha) = q(\alpha)(\cancel{\alpha-\alpha}^{\,0}) + r = r.$$

We conclude that $(x-\alpha)$ divides $f(x)$ if and only if $f(\alpha)=0$.

$\blacksquare$

This finally justifies our method of solving cubic equations. Let $f(x)$ be a polynomial of degree 3.

- Guess a solution $f(r)=0$.

- Divide $f(x)$ by $(x-r)$ to obtain

$$f(x) = g(x)(x-r) + 0.$$

Note that $\deg(g)=2$ because

$$\begin{aligned} 3 &= \deg(f(x)) \\ &= \deg(g(x)(x-r)) \\ &= \deg(g(x)) + \deg((x-r)) \\ &= \deg(g(x)) + 1 \end{aligned}$$

- Solve $g(x) = 0$ using the Quadratic Formula.

The remaining weakness of this method is in the first step:

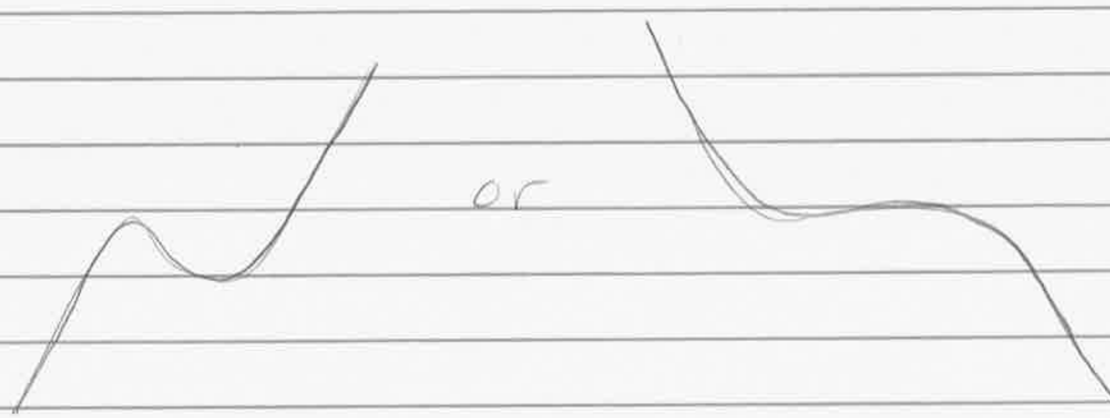
Q: What if we can't guess a solution?

There are two ways to proceed.

1. Practical
2. Theoretical.

Let's do 1 First.

Consider a cubic polynomial with real coefficients $f(x) \in \mathbb{R}[x]$. Its graph looks like



In either case, the graph must cross the x-axis somewhere.

Theorem: Suppose $f(x) \in \mathbb{R}[x]$ has degree 3. Then the equation $f(x) = 0$ has at least one real solution.

Proof: Use the intermediate value theorem (see MTH 433).

So we are guaranteed that a real solution exists. We can compute it to any desired degree of accuracy using something like Newton's Method.

Example: Find an approximate real solution to the cubic

$$x^3 - x + 2 = 0.$$

[You won't be able to guess a solution this time!]

1/28/15

HW 1 due Friday
Office Hours: Mon 1-2 & Wed 2-3.

We are trying to solve cubic equations

$$ax^3 + bx^2 + cx + d = 0.$$

If we can't <u>guess</u> a solution, then there are two ways to proceed:

1. Practical
   - Try to compute an approximate solution

2. Theoretical
   - Try to find an "algebraic formula" for the solution.

We'll do 1 first because it's easier.

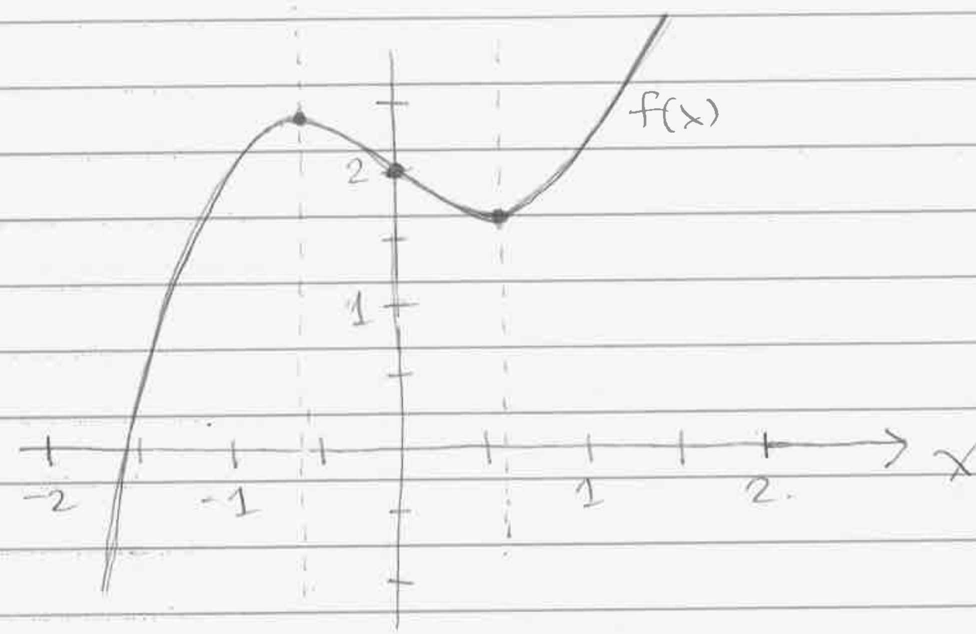
<u>Example</u>: Solve

$$x^3 - x + 2 = 0.$$

[I'm willing to bet you won't be able to guess a solution!]

Every real cubic has a real root (by Intermediate Value Theorem), so we know that there exists at least one solution.

We can graph the function $f(x) = x^3 - x + 2$.



This suggests that there is one real solution, around $x = -1.5$.

There is a very effective method to approximate this solution called

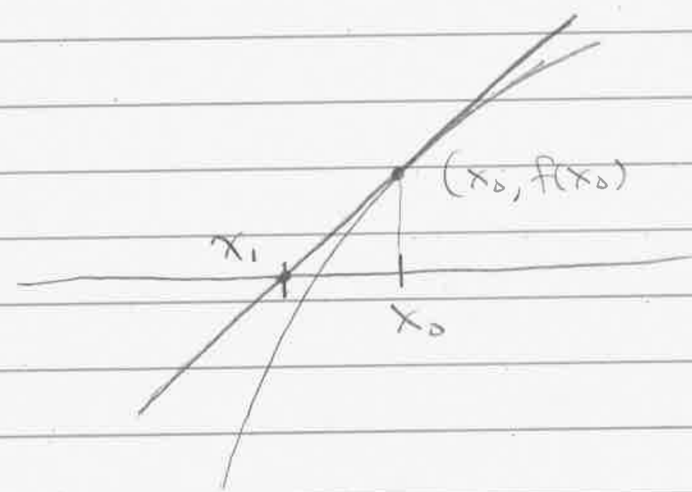
"Newton's Method".

Newton's Method:

- Guess a solution to $f(x) = 0$.

[We guess $x_0 = -1.5 = -3/2$]

- Compute the tangent line to graph of $f(x)$ at the point $(x_0, f(x_0))$. Let x_1 be the x-intercept of this line.



The tangent line has slope $f'(x_0)$ and passes through the point $(x_0, f(x_0))$, so it has equation

$$(y - f(x_0)) = f'(x_0)(x - x_0).$$

Let $y = 0$ to find the x-intercept:

$\downarrow$

$$-f(x_0) = f'(x_0)(x - x_0)$$

$$x - x_0 = -f(x_0)/f'(x_0)$$

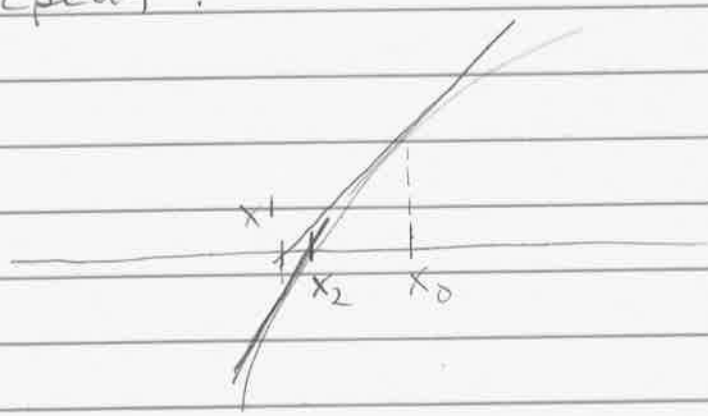
$$x = x_0 - f(x_0)/f'(x_0)$$

So we define

$$x_1 := x_0 - f(x_0)/f'(x_0).$$

[The idea is that x_1 is a better approximation to a solution.]

- Now repeat.



Define $x_{n+1} := x_n - f(x_n)/f'(x_n)$.

[We expect that $x_0, x_1, x_2, x_3, \ldots$ will converge to a solution.]

Kantorovich's Theorem (1940) guarantees that Newton's Method almost always converges, and when it does it converges very quickly.

In our example we have

$$f(x) = x^3 - x + 2$$
$$f'(x) = 3x^2 - 1$$

so the iteration formula is

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$
$$= x_n - \frac{x_n^3 - x_n + 2}{3x_n^2 - 1}.$$

We guess $x_0 = -3/2$. We compute

$$x_1 = -4243/2783 \approx -1.521379806\cdots$$

$$x_2 \approx -1.521379707\cdots$$

$$x_3 \approx -1.521379707\cdots$$

That's good enough for me! ///

This gives a method to solve cubic equations $f(x)=0$ approximately:

- Use Newton's Method to get an approximate solution $f(r) \approx 0$.

- Divide $f(x)$ by $(x-r)$ to get

$$f(x) = (x-r)\,g(x) + \varepsilon$$

where $\varepsilon \approx 0$.

- The solutions of $g(x)=0$ are approximate solutions to $f(x)=0$.

[Remark: This method can be iterated to solve polynomial equations of any degree.]

Is that good enough? NO!

We want to have an exact algebraic expression for the solutions.

Why?

- Aesthetic / Religious reasons.
- We know that we will learn important things from the formula, just as we learned important things from the Quadratic Formula.

The Cubic Formula.

To proceed we will need a trick analogous to "completing the square". This trick was unknown in the ancient world; it was discovered in Italy in the 1500s.

The Trick: If $x = u + v$, then

$$x^3 - 3uv\,x - (u^3 + v^3) = 0.$$

Stay tuned to see how this is used to solve the cubic...