Math 461 **Spring 2015**
Homework 2 **Drew Armstrong**

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1. Let $\mathbb{F}$ be a field and consider two polynomials $f(x), g(x) \in \mathbb{F}[x]$.

(a) If $f(x)$ and $g(x)$ are both nonzero, prove that $\deg(fg) = \deg(f) + \deg(g)$.

(b) How should you define the "degree" of the zero polynomial so that the result in part (a) remains true even when one or both of $f(x)$ and $g(x)$ is zero?

2. Let $\mathbb{F}$ be a field with **finitely** many elements. Prove that there must exist two non-equal polynomials (i.e., with different coefficients) that yield equal functions $\mathbb{F} \to \mathbb{F}$. [Hint: How many different polynomials are there? How many different functions?]

3. Let $\mathbb{F}$ be a field and consider the ring of polynomials $\mathbb{F}[x]$. Apply Descartes' Factor Theorem to prove the following statement: If $f(x) \in \mathbb{F}[x]$ has degree n, then $f(x)$ has **at most** n distinct roots in $\mathbb{F}$. [Hint: Use induction.]

4. Assume that the cubic equation $ax^3 + bx^2 + cx + d = 0$ has three distinct roots, called r, s, t. Give a formula for $rs + rt + st$ in terms of the coefficients a, b, c, and d.

5. Prove that $\sqrt[3]{7+\sqrt{50}} + \sqrt[3]{7-\sqrt{50}} = 2$. [Hint: Maybe the cube roots of $7+\sqrt{50}$ and $7-\sqrt{50}$ have the form $a + b\sqrt{2}$, where a and b are small whole numbers.]

6. Define a function $f : \mathbb{C} \to \mathrm{M}_{2\times 2}(\mathbb{R})$ from complex numbers to real 2×2 matrices by setting

$$f(a+ib) := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

For any complex numbers $u, v \in \mathbb{C}$ verify the following:

(a) $f(u+v) = f(u) + f(v)$

(b) $f(uv) = f(u)f(v)$

(c) $|u|^2 = \det f(u)$.

(The operations on the right hand side are matrix addition, matrix multiplication, and matrix determinant.)