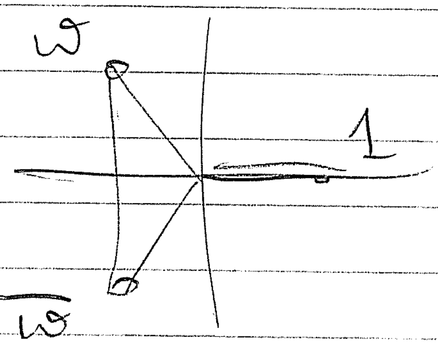
HW 4 due Fri

Typo: $x = 2\cos(2\pi/7)$
satisfies
$$x^3 + x^2 - 2x - 1 = 0.$$

Office Hours
Wed & Thurs. 2:30 – 4:00

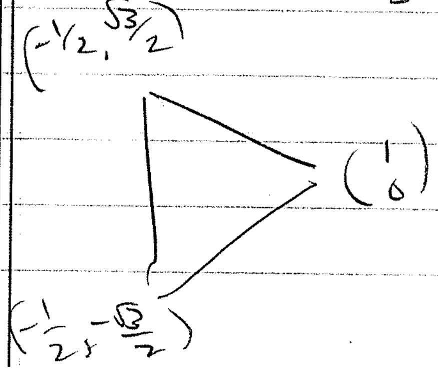
Today: Regular Polygons.

We saw: Regular triangle is constructible



$$\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

$$1 + \omega + \overline{\omega} = 0$$
$$\omega + \overline{\omega} = -1$$
$$2\cos\left(\frac{2\pi}{3}\right) = -1$$
$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \implies \sin\left(\frac{2\pi}{3}\right) = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}.$$



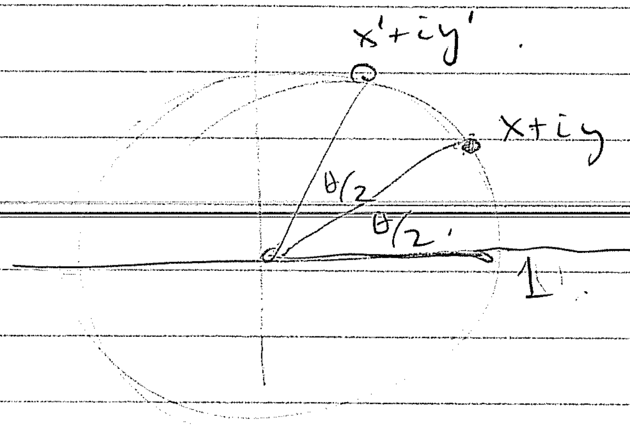
Constructible

Regular hexagon constructible.

In fact, we have

n-gon c'ble $\Leftarrow\Rightarrow$ $\frac{n}{2}$-gon c'ble.

__Proof__. Consider.



de Moivre.

$$x'+iy' = (x+iy)^2$$
$$= (x^2-y^2) + i(2xy).$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x^2-y^2 \\ 2xy \end{pmatrix}.$$

So (x,y) c'ble $\Rightarrow$ (x',y') c'ble.

Conversely, since $x^2+y^2=1$ we can write,

$$x' = x^2 - (1 - x^2) = 2x^2 - 1$$

$$\Rightarrow \quad x^2 = \frac{1 + x'}{2}$$

$$x = \pm\sqrt{\frac{1 + x'}{2}}$$

And $$y^2 = 1 - x^2 = 1 - \left(\frac{1 + x'}{2}\right)$$

$$= \frac{2 - (1 + x')}{2} = \frac{1 - x'}{2}$$

$$y = \pm\sqrt{\frac{1 - x'}{2}}.$$

Hence

x', y' c'ble $\Rightarrow$ x, y c'ble $\blacksquare$

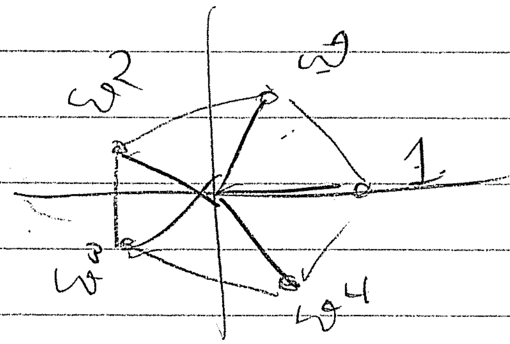
Know. $2, 3, 4, 6, 8, 12, 16, \cdots$ gons c'ble

What else? 5?

Theorem: 5-gon is c'ble.

(Known to Greeks).

Consider



$\omega = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$

Let $A = \omega + \omega^4 = \omega + \overline{\omega} > 0$

$B = \omega^2 + \omega^3 = \omega^2 + \overline{\omega^2} < 0$

Then $A + B = \omega + \omega^2 + \omega^3 + \omega^4 = -1$

$AB = (\omega + \omega^4)(\omega^2 + \omega^3)$

$= \omega^3 + \omega^6 + \omega^4 + \omega^7$

$= \omega^3 + \omega + \omega^4 + \omega^2 = -1.$

Hence A, B are solutions of

$$(x-A)(x-B) = x^2 - (A+B)x + AB.$$
$$= x^2 + x - 1$$

$$A, B = \frac{-1 \pm \sqrt{1^2+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

which is which?

$$A = \frac{-1+\sqrt{5}}{2} > 0.$$

Hence $\omega + \overline{\omega} = \frac{-1+\sqrt{5}}{2}$

$$2\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{2}$$

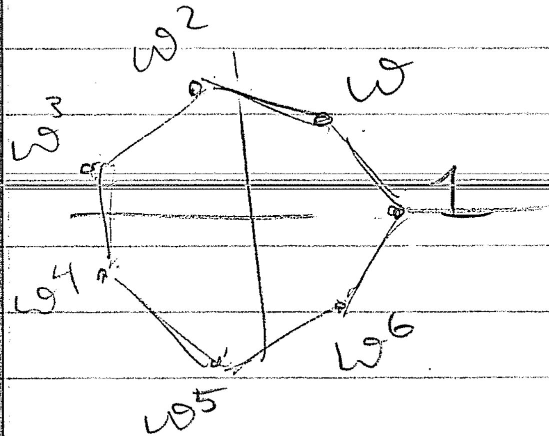
So $\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}$. c'ble!

Use de Moivre to show $1, \omega, \omega^2, \omega^3, \omega^4$ are all c'ble points

$\blacksquare$

Next: the 7-gon.

$\omega = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$



Goal: Solve for $2\cos\left(\frac{2\pi}{7}\right) = \omega + \omega^6 = \underline{\omega + \overline{\omega}}$.

Well...

$$0 = 1 + \omega + \omega^2 + \cdots + \omega^6.$$
$$= \omega^3 + \omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} + \omega^{-3}.$$

Let $\omega + \omega^{-1} = u$.

$$u^3 = \omega^3 + 3\omega^2\omega^{-1} + 3\omega\omega^{-2} + \omega^{-3}$$
$$= \omega^3 + 3\omega + 3\omega^{-1} + \omega^{-3}$$

$$u^2 = (w+w^{-1})^2 = w^2 + 2ww^{-1} + w^{-2}$$
$$= w^2 + 2 + w^{-1}$$

$$u^1 = w + w^{-1}.$$

$$\begin{array}{rl|l}
\text{Hence} & u^3 = & w^3 + 0 + 3w + 0 + 3w^{-1} + 0 + w^{-3} \\
& + u^2 & w^2 + 0 + 2 + 0 + w^{-2} \\
& -2u & -2w + 0 - 2w^{-1} \\
& -1 & -1 \\
\hline
& 0 & w^3 + w^2 + w + 1 + w^{-1} + w^{-2} + w^{-3} = 0
\end{array}$$

$u = 2\cos\left(\frac{2\pi}{7}\right)$ is a root of

$$u^3 + u^2 - 2u + 1 = 0.$$

$\underset{\text{HW4}}{\Longrightarrow}$ NOT constructible

What went wrong?

$7-1=6$ is not a power of 2!

In 1796 it was known.

- n-gon c'ble $\Rightarrow$ 2n-gon c'ble.
- triangle, pentagon c'ble.
- heptagon ???

Then Gauss (age 19) came along.

Theorem (Gauss, 1796).

$$16\cos\left(\frac{2\pi}{17}\right) = -1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} +$$
$$2\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}.$$

Hence 17-gon (!) is c'ble. WHAT!?

In fact, he did more...

Thm: If $n = 2^m p_1 p_2 \cdots p_k$

where $p_1 < p_2 < \cdots < p_k$ primes with

$p_i - 1 =$ power of 2 "Fermat primes"

Then n-gon is c'ble (!)

eg. 257-gon (!)

(Wantzel, 1837) showed only if.

e.g. 7, 9, 11, ⋯ not c'ble.

~~Def~~ $\cancel{F_m = 2^{2^m} + 1}$

If p prime, $p - 1 = 2^a$,

- p is called a "Fermat" prime.
- $a = 2^m$ (HW 3).

Let $F_m = 2^{2^m} + 1$ (m^{th} "Fermat number")

$F_0 = 3$, $F_3 = 257$

$F_1 = 5$, $F_4 = 65537$ all prime.

$F_2 = 17$

Fermat conjectured F_m prime $\forall m$.

Euler (1732) showed

$F_5 = 641 \cdot 6700417$ NOT prime.

No other Fermat prime has ever been found.

HW 4 due Fri ⇝ check latest version on the web.

Office Hours

Today & Tomorrow

2:30 – 4:00.

Before 1796 it was known

- n-gon c'ble ⇒ $2n$-gon c'ble
- triangle, pentagon c'ble
- heptagon ???

Then Gauss (age 19) came along

Thm (Gauss).

If $n = 2^m p_1 p_2 p_3 \cdots p_k$

where

$p_1 < p_2 < p_3 < \cdots < p_k$ "Fermat" primes

such that $p_i - 1 = $ power of 2 $\forall i$.

Then regular n-gon is c'ble.

(he didn't say how).

eg. 257 is a Fermat prime

⟹ 257-gon is c'ble (!)

Wantzel (1837) showed only if.

eg. 7 is prime
$7-1$ not a power of 2
$\Rightarrow$ 7-gon is NOT c'ble.

If p is prime and $p-1=2^a$,
Then p is called a Fermat prime.

Thm: In this case, $a=2^m$ for some m.

Proof: suppose $2^a+1=p$ is prime and a NOT a power of 2. Then $a=nb$ for some odd $b\neq 1$. Consider difference of bth powers

$$1-x^b=(1-x)(1+x+x^2+\cdots+x^{b-1}).$$

Since b is odd put $x\to -x$ to get

$$1+x^b=(1+x)(1-x+x^2-x^3+\cdots+x^{b-1})$$

Now put $x\to 2^n$ to get

$$p=1+2^a=1+(2^n)^b=(1+2^n)(1-2^n+2^{2n}-2^{3n}+\cdots+2^{(b-1)n})$$

$\Rightarrow$ p is not prime $\lightning$ $\boxtimes$

Hence $a=2^m$ for some m.

"Mersenne prime"

[Exercise: If $p = 2^a - 1$ is prime
Then a is prime.]

Let $F_m = 2^{2^m} + 1$ = m^{th} "Fermat number"

$F_0 = 3$ $\quad F_3 = 257$
$F_1 = 5$ $\quad F_4 = 65537$
$F_2 = 17$:)

all prime.

Fermat conjectured F_m prime $\forall m$.

Euler showed.

$F_6 = 641 \cdot 6700417$ NOT prime.

In fact:
- no other Fermat prime has been found!
Fermat could not have been more wrong.

Summary:

$\cos\left(\frac{\pi}{n}\right)$ has nice formula $\Leftrightarrow$ Regular n-gon c'ble $\Leftrightarrow$ $n = 2^m p_1 \cdots p_k$.

We showed earlier that

$u^3 - 3u - 1 \in \mathbb{Q}[u]$ has no root in $\mathbb{Q}$.

How'd we do it?

Thm (Rational Root Test).

Consider.

$f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \in \mathbb{Q}[x]$

If $x = \frac{a}{b} \in \mathbb{Q}$ is a root.
($a, b \in \mathbb{Z}$, a, b coprime).

Then a divides c_0
b divides c_n.

Proof: Plug in $x = a/b$

$$c_n \frac{a^n}{b^n} + c_{n-1} \frac{a^{n-1}}{b^{n-1}} + \cdots + c_1 \frac{a}{b} + c_0 = 0.$$

Multiply by b^n.

$$c_n a^n + c_{n-1} a^{n-1} b + \cdots + c_1 a b^{n-1} + c_0 b^n = 0.$$

Moving $c_0 b^n$ to RHS $\Longrightarrow$
a divides $c_0 b^n$.

$a \mid (c_0 b^{n-1}) b$ and a, b coprime

$\overset{\text{HW4}}{\Longrightarrow} \quad a \mid c_0 b^{n-1} = (c_0 b^{n-2}) b$

$\Rightarrow \quad a \mid c_0 b^{n-2}$

$\vdots$

$\Rightarrow \quad a \mid c_0$.

Similarly, taking $c_n a^n$ to RHS

$\Rightarrow \quad b \mid c_n a^n \overset{\text{HW4}}{\Longrightarrow} \quad b \mid c_n$ $\boxed{\;}$

eg. Consider

$$3x^3 - 5x^2 + 5x - 2 = 0.$$

If $x = \frac{a}{b} \in \mathbb{Q}$ is a root then

- $a \mid 2 \Rightarrow a = \pm 1, \pm 2$
- $b \mid 3 \Rightarrow b = \pm 1, \pm 3$

So $\frac{a}{b} = \frac{\pm 1, 2}{\pm 1, 3} = \underbrace{\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}}_{8 \text{ possibilities.}}$

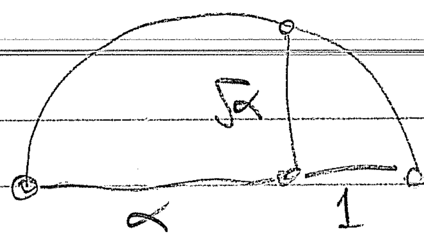
Plugging in we find $+\frac{2}{3}$ is the only $\mathbb{Q}$-root.

HW 4 due now.

Exam 2, Fri Mar 25.

Topics:

- $\sqrt{2} \notin \mathbb{Q}$
- straightedge and compass
  $1, +, -, \times, \div, \sqrt{\ }$ $\Rightarrow$ constructible.



- constructible $\Rightarrow$ $1, +, -, \times, \div, \sqrt{\ }$
- Quadratic Field Extension
  $F \subseteq F[\sqrt{c}]$
- α constructible $\Longleftrightarrow$ $\exists\ \mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq \underset{\alpha}{\overset{}{F_k}}$ ($\alpha \in F_k$)

- Rational Root Test.
- Lemma: for cubic $f(x) \in F[x]$, $F \subseteq F[\sqrt{c}]$,
Then,
f has root in $F[\sqrt{c}]$ $\Rightarrow$ f has root in F.
- Impossible constructions
$u^3 - 2$ $\Rightarrow$ $\sqrt[3]{2}$ NOT c'ble $\Rightarrow$ can't double the cube
$u^3 - 3u - 1$ $\Rightarrow$ $\cos(\frac{\pi}{9})$ NOT c'ble $\Rightarrow$ can't trisect angle.
$u^3 + u^2 - 2u - 1$ $\Rightarrow$ $2\cos(\frac{2\pi}{7})$ NOT c'ble $\Rightarrow$ can't construct 7-gon

Now: Special Topic $\mathbb{C}$ vs. $\mathbb{Q}[\sqrt{2}]$

In $\mathbb{C}$:

$$\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} \approx (a+ib)(c+id) = (ac-bd)+i(ad+bc)$$

$$\approx \begin{pmatrix} ac-bd \\ ad+bc \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$$

$a+ib$ <u>acts like</u> $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

In $\mathbb{Q}[\sqrt{2}]$:

$$(a+b\sqrt{2})(c+d\sqrt{2}) = (ac+2bd)+\sqrt{2}(ad+bc)$$

$$\approx \begin{pmatrix} ac+2bd \\ ad+bc \end{pmatrix} = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$$

$a+b\sqrt{2}$ <u>acts like</u> $\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$

In $\mathbb{C}$: $z \to z\bar{z}$

"norm"

$$= N(a+ib)$$

$$(a+bi)(a-bi) = a^2 + b^2 = (\text{"length"})^2$$

In $\mathbb{Q}[\sqrt{2}]$: $v \to v\bar{v}$

"norm"

$$(a+b\sqrt{2})(a-b\sqrt{2}) = a^2 - 2b^2 = ? \text{ call it } N(a+b\sqrt{2})$$

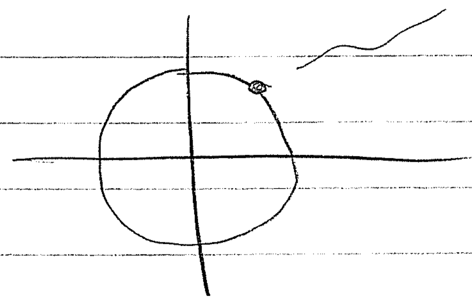
We have $N(uv) = uv\overline{uv} = uv\bar{u}\bar{v}$

$$= u\bar{u}v\bar{v} = N(u)N(v).$$

multiplicative.

<u>Unit</u> circle in $\mathbb{C}$:

$x+iy$ where $N(x+iy) = x^2+y^2 = 1$

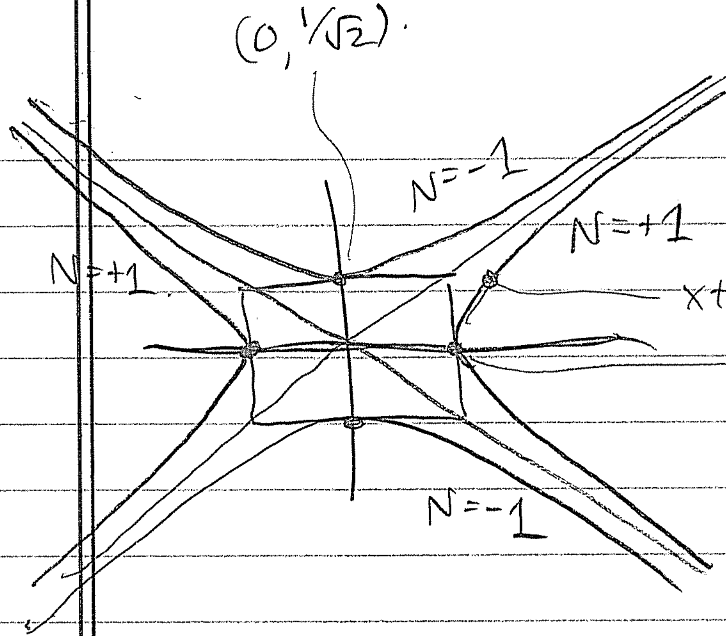


"Unit circle" in $\mathbb{Q}[\sqrt{2}]$?

$$N(x+y\sqrt{2}) = x^2 - 2y^2 = 1$$

$$\left(\frac{x}{1}\right)^2 - \left(\frac{y}{1/\sqrt{2}}\right)^2 = 1$$

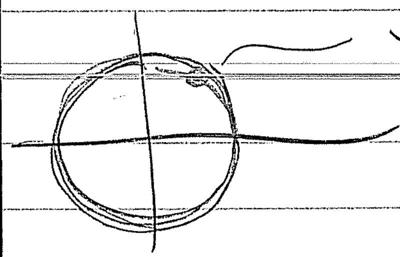
<u>hyperbola</u>.



"unit circle"

$x+y\sqrt{2}$ where $x^2-2y^2=+1$

De Moivre in $\mathbb{C}$.



$x+iy = \cos\theta + i\sin\theta$.

Then $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$.

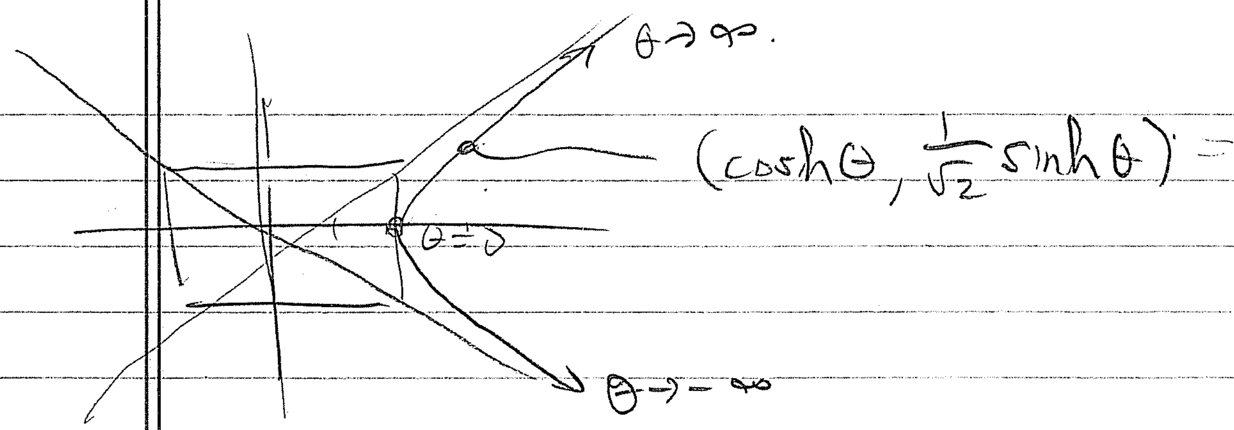
"multiply points" = "add angles".

de Moivre in $\mathbb{Q}[\sqrt{2}]$?

$x^2-2y^2=+1 \implies \exists\, \theta$ s.t. $x=\cosh\theta$, $y=\frac{1}{\sqrt{2}}\sinh\theta$.

$\cosh\theta = \cos(i\theta) = \dfrac{e^{\theta}+e^{-\theta}}{2}$

$\sinh\theta = \sin(i\theta) = (e^{\theta}-e^{-\theta})/2$



$$\left((\cosh\alpha) + \left(\tfrac{1}{\sqrt{2}}\sinh\alpha\right)\sqrt{2}\right)\left((\cosh\beta) + \left(\tfrac{1}{\sqrt{2}}\sinh\beta\right)\sqrt{2}\right)$$

$$= \left[\cosh(\alpha+\beta)\right] + \left[\tfrac{1}{\sqrt{2}}\sinh(\alpha+\beta)\right]\sqrt{2}.$$

"multiply points" = add "angles".
θ parameter

Application: Find integer solutions to $x^2 - 2y^2 = 1$. (Pell's Equation).

Note. $\binom{x}{y} = \binom{3}{2}$ works. i.e. $N(3+2\sqrt{2}) = 1$.

But then $N\left((3+2\sqrt{2})(3+2\sqrt{2})\right) = 1 \cdot 1 = 1$.

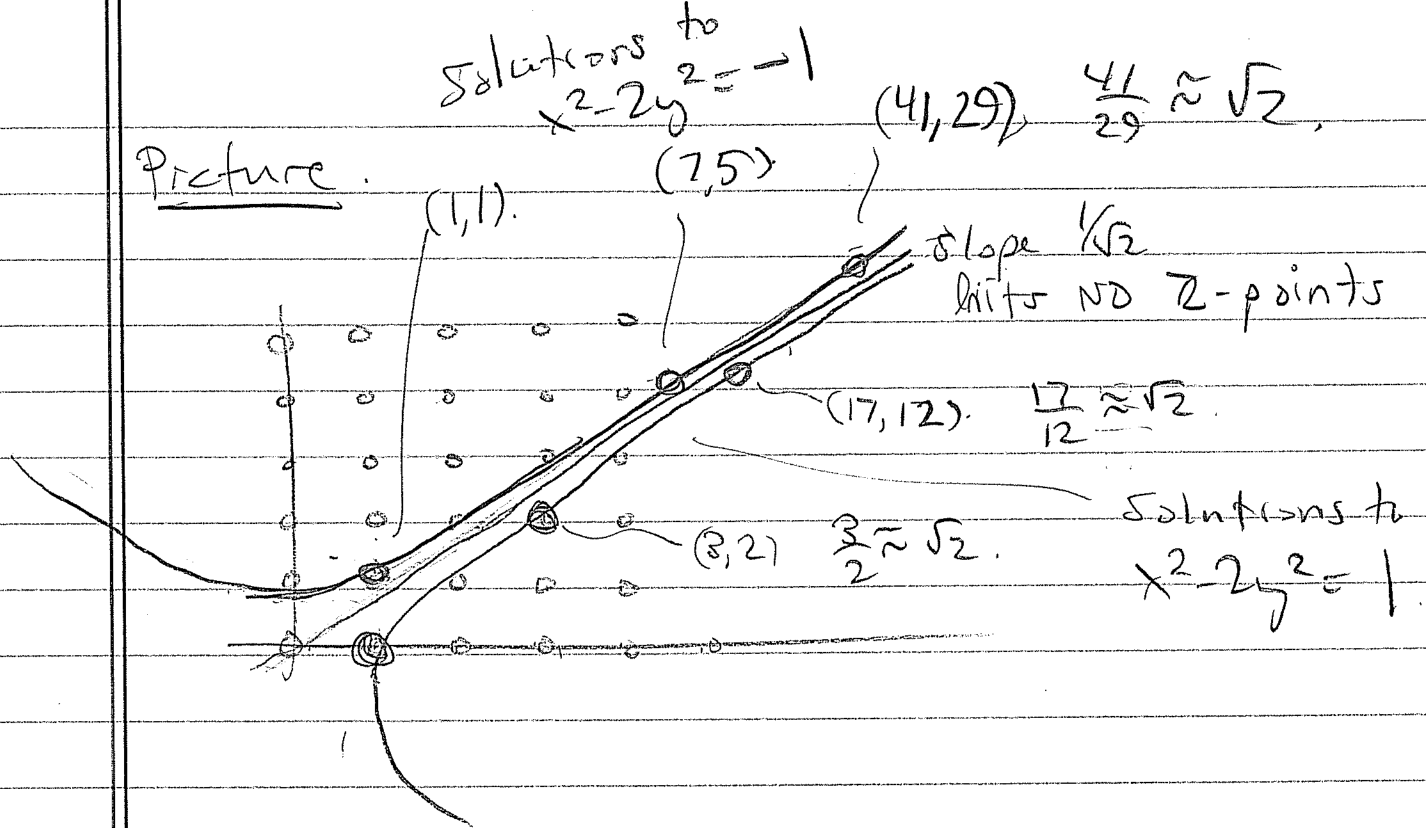
So $(9+8) + (6+6)\sqrt{2} = 17 + 12\sqrt{2}$.

$\Longrightarrow \binom{17}{12}$ another solution

Then $(3+2\sqrt{2})(17+12\sqrt{2}) = 99 + 70\sqrt{2}$.

$\Longrightarrow \binom{99}{70}$ another solution. etc.

Picture.



Pell's Equation $\Longleftrightarrow$ Rational Approximations to $\sqrt{2}$.

$\Updownarrow$

"prefixes" of

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}$$

$$\sqrt{2} = 1+\sqrt{2}-1 = 1+\frac{1}{\sqrt{2}+1} = 1+\frac{1}{2+\sqrt{2}-1}$$

$$= 1+\cfrac{1}{2+\cfrac{1}{\sqrt{2}+1}} \quad \text{etc.}$$

ATTN: Mcknight/Zame Lecture
5pm Tonight Wesley Gallery.
"Plane Tilings"

HW 3 Average 18/20
HW 4 Average 23.4/24.

Exam 2 Friday
- same format as Exam 1.

Office Hours.
Wed & Thurs 2:30 — 4:00

This week review.

Roots come in conjugate pairs.

Consider QFE $\mathbb{R} \subseteq \mathbb{R}[\sqrt{-1}] = \mathbb{C}$.

~~If $f(\alpha) = 0$ with $f(x) \in \mathbb{R}[x]$ and $\alpha \in \mathbb{C}$.~~
~~Then $f(\bar{\alpha}) = 0$ also.~~

Suppose Say $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$
$a_0, a_1, \ldots, a_n \in \mathbb{R}$.

with $f(\alpha) = 0$ for some $\alpha \in \mathbb{C}$.

$\mathbb{R} \subseteq \mathbb{C}$ has a conjugation map.

$$\overline{a+ib} = a-ib.$$

Note: for $z \in \mathbb{C}$ we have

$$z = \overline{z} \iff z \in \mathbb{R}.$$

Proof: Let $z = a+ib$.
If $z \in \mathbb{R}$ then $b=0$ hence $z = \overline{z} = a$.
Conversely, if $z = \overline{z}$ then $a+ib = a-ib$.
$\Rightarrow 2ib = 0 \Rightarrow b = 0 \Rightarrow z \in \mathbb{R}$ $\boxtimes$

Now consider $f(x) \in \mathbb{R}[x]$. Say

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

If $f(\alpha) = 0$ for some $\alpha \in \mathbb{C}$ then

$$\overline{a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0} = \overline{0}$$

$$\overline{a_n}(\overline{\alpha})^n + \overline{a_{n-1}}(\overline{\alpha})^{n-1} + \cdots + \overline{a_1}\,\overline{\alpha} + \overline{a_0} = \overline{0}$$

$$a_n(\overline{\alpha})^n + a_{n-1}(\overline{\alpha})^{n-1} + \cdots + a_1\overline{\alpha} + a_0 = 0$$

$$f(\overline{\alpha}) = 0.$$

So for any $\alpha \in \mathbb{C}$, $f(\alpha) = 0 \Rightarrow f(\bar{\alpha}) = 0$.

Letting $\alpha = \bar{\beta}$ we get.

$$f(\bar{\beta}) = 0 \Rightarrow f(\bar{\bar{\beta}}) = 0$$

$$\Rightarrow f(\beta) = 0. \quad \forall \beta \in \mathbb{C}.$$

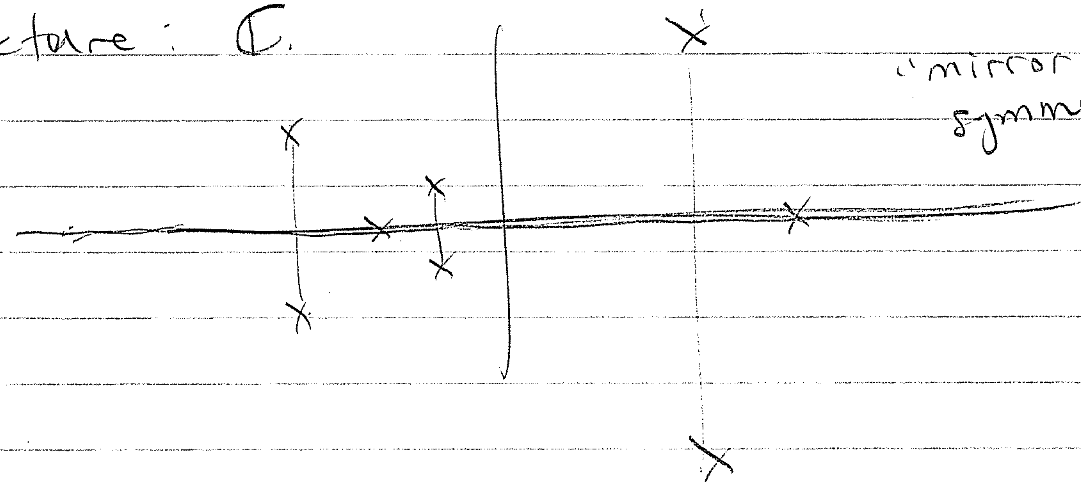
Hence

$$f(\alpha) = 0 \Leftrightarrow f(\bar{\alpha}) = 0$$

$\mathbb{C}$-roots of $\mathbb{R}$-polynomials come in conjugate pairs.

[In general, for QFE $F \subseteq F[\sqrt{c}]$, $F[\sqrt{c}]$-roots of F-polynomials come in conjugate pairs]

Picture: $\mathbb{C}$.



"mirror symmetry".

X's = roots of some deg 8 poly $\in \mathbb{R}[x]$

Corollary: Let $f(x) \in \mathbb{R}[x]$ be cubic.

If f has a $\mathbb{C}$-root then it has an $\mathbb{R}$-root.

Proof: Suppose $f(\alpha) = 0$ for $\alpha \in \mathbb{C}$.
If $\alpha \in \mathbb{R}$, done. So suppose $\alpha \in \mathbb{R} \setminus \mathbb{C}$.
Then $\overline{\alpha} \neq \alpha$ is another $\mathbb{C}$-root.
By factor theorem.

$$f(x) = (x - \alpha)(x - \overline{\alpha})\, g(x)$$

where $g(x) \in \mathbb{R}[x]$ has degree 1.
Say

$$g(x) = ax + b \quad \text{for } a, b \in \mathbb{R},\ a \neq 0.$$

But then,

$$g(b/a) = 0 \implies f(b/a) = 0$$

$\implies \frac{b}{a}$ is a $\mathbb{R}$-root of $f(x)$

Similarly, given $F \subseteq F[\sqrt{c}]$, cubic $f(x) \in F[x]$,

$f(x)$ has $F[\sqrt{c}]$-root $\implies$ $f(x)$ has F-root.

Q: Does it work for higher degree f?

Sp. $f(x) \in F[x]$ has degree 5.
$f(\alpha) = 0$ for some $\alpha \in F[\sqrt{c}]$.
If $\alpha \in F$ done.
Otherwise $\alpha, \bar{\alpha}$ are two roots.
Use Factor Theorem

$$f(x) = (x - \alpha)(x - \bar{\alpha})\, g(x)$$

where $g(x) \in F[x]$ has degree 3.

If $g(x)$ has a $F[\sqrt{c}]$-root we can repeat.
But does it?

Problem: In general we don't know *if* polynomials have roots

$g(x) \in F[x]$ might have 0 roots in $F[\sqrt{c}]$.
Then we're stuck. :(

[Luckily, EVERY $f(x) \in \mathbb{R}[x]$ has a root in $\mathbb{C}$.
Deep Result.]

Exam 2 Friday

OH Today & Tomorrow 2:30 – 4:00

Today: Review.

$\sqrt{2} \notin \mathbb{Q}$ CRISIS

Greeks abandoned numbers for straightedge-and-compass.

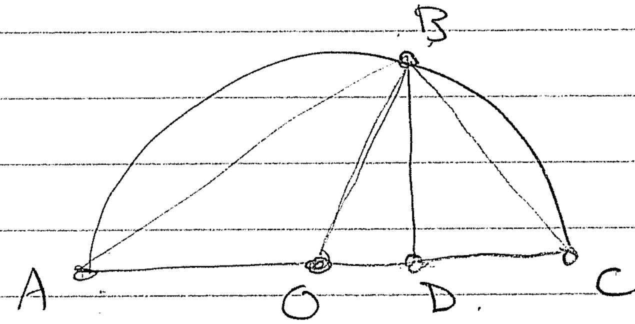
- 2 points $\rightarrow$ line
- 2 points $\rightarrow$ circle
- new points from intersections line $\cap$ line, line $\cap$ circle, circle $\cap$ circle.

Starting with $\underbrace{\circ\!\!-\!\!\circ}^{\text{"1"}}$, what lengths can you construct?

Thm: If α, β are c'ble then so are

- $\alpha \pm \beta$
- $\alpha\beta$, α/β $\Big\}$ a field.
- $\sqrt{\alpha}$.

eg.



Let $AD = \alpha$

$DC = 1$

Claim $BD = \sqrt{\alpha}$.

Proof: first show $\angle ABC = 90^\circ$.

Let $\angle OAB = \angle OBC = \theta$ (isoceles)

$\angle OBC = \angle OCB = \mu$ (isoceles).

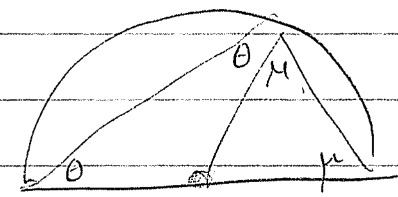
Then

$$\theta + (\theta + \mu) + \mu = 180^\circ \quad (\text{in } \Delta ABC)$$

$$\Rightarrow \quad 2(\theta+\mu) = 180^\circ$$

$$\theta + \mu = 90^\circ$$

$$\angle ABC = 90^\circ \quad \checkmark$$



Then note ΔADB similar to ΔBDC (same angles).

Hence. $$\frac{AD}{BD} = \frac{BD}{DC}.$$

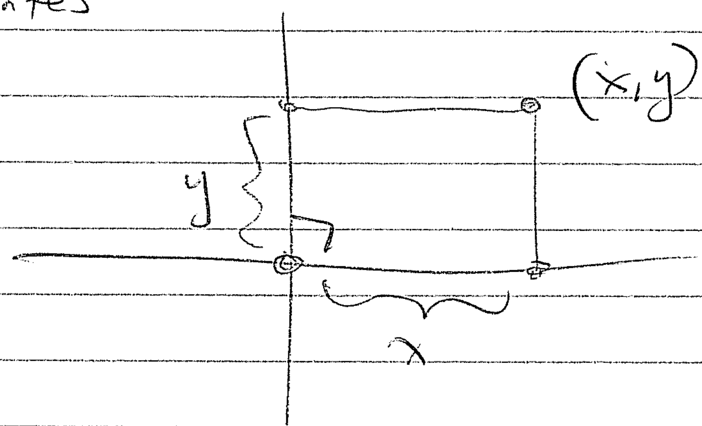
$$\frac{\alpha}{BD} = \frac{BD}{1}$$

$$BD = \sqrt{\alpha}.$$

α c'ble $\Rightarrow$ $\sqrt{\alpha}$ c'ble.

1637: Descartes / Fermat rescued numbers

coordinates



point (x,y) c'ble $\iff$ lengths x, y c'ble.

How to get new c'ble points?

Intersect $\underbrace{ax+by+c=0}_{\text{lines}}$ & $\underbrace{(x-a')^2+(y-b')^2=c'}_{\text{circles}}$

with a,b,c,a',b',c' c'ble.

At worst the solutions look like $\frac{A \pm \sqrt{B}}{C}$

with A, B, C c'ble.

So c'ble #'s $\subseteq$ / $\supseteq$ $\overbrace{1, +, -, \times, \div}^{\text{Field}}$, $\sqrt{\ }$

More formally, $\alpha \in \mathbb{R}$ is c'ble $\iff \exists$

$$\underbrace{\mathbb{Q} \subseteq F_1 \subseteq F_2 \subseteq \cdots}_{\text{quadratic extensions.}} \subseteq F_k \ni \alpha$$

Then we can <u>prove</u> things.

e.g. $\sqrt[3]{2}$, $\cos\left(\frac{\pi}{9}\right)$, $\cos\left(\frac{2\pi}{7}\right)$
NOT c'ble.

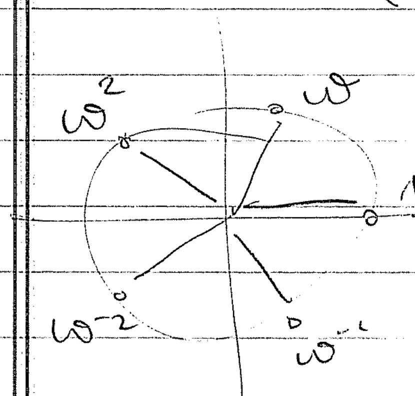
Gauss / Wantzel found.

$$\cos\left(\frac{\pi}{n}\right) \text{ c'ble} \iff n = 2^k p_1 p_2 \cdots p_r$$

where $p_1 < p_2 < \cdots < p_r$ are Fermat primes
$$p_i = 2^{2^{e_i}} + 1$$

<u>Eg.</u> $5 = 2^{2^1} + 1$ is Fermat

$\Longrightarrow \cos\left(\frac{\pi}{5}\right)$ <u>is</u> c'ble. Let's see ...



let $\omega = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$

So
$$\omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} = 0$$

Want $u = \omega + \omega^{-1} = 2\cos\left(\frac{2\pi}{5}\right)$.

Note: $u^2 = (\omega + \omega^{-1})^2 = \omega^2 + 2\overset{1}{\overset{\omega^0}{\cancel{\omega\omega^{-1}}}} + \omega^{-2}$
$= \omega^2 + 2 + \omega^{-1}$.

So $u^2 + u - 1 = \omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} = 0$

Hence $u = \dfrac{-1 \pm \sqrt{5}}{2} = \dfrac{-1+\sqrt{5}}{2} > 0$

$$2\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{2}$$

$$\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}.$$

$$\Longrightarrow \quad \cos\left(\frac{\pi}{5}\right) = \sqrt{\frac{1+\cos\left(\frac{2\pi}{5}\right)}{2}}$$

$$= \sqrt{\frac{1}{2} + \frac{\sqrt{5}-1}{8}} = \sqrt{\frac{4+\sqrt{5}-1}{8}}$$

$$= \sqrt{\frac{3+\sqrt{5}}{8}} \qquad \text{c'ble}$$