HW 3 due March 4
(not posted yet)
Exam 2 March 25.

<u>Today</u>: Greek

The Pythagoreans (~ -500)
- "all is number"
- "number" = postive integers and their ratios

<u>The Crisis</u>: $\sqrt{2}$ is not a "number".

<u>Proof</u>: Suppose $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.
Square to get $2 = a^2/b^2$, or $2b^2 = a^2$.
Since a^2 is <u>even</u>, a is even, say $a = 2a'$.
Then $2b^2 = (2a')^2 = 4(a')^2$, or $b^2 = 2(a')^2$.
Since b^2 is <u>even</u>, b is <u>even</u>, say $b = 2b'$.

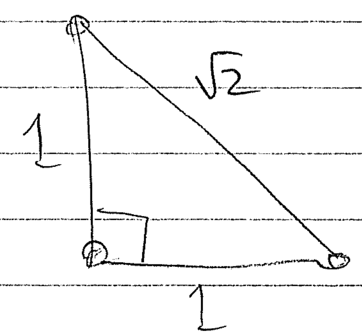
We get $\sqrt{2} = \dfrac{a}{b} = \dfrac{2a'}{2b'} = \dfrac{a'}{b'}$ with $a > a' \geq 1$, $b > b' \geq 1$.

Repeat to get $\sqrt{2} = \dfrac{a}{b} = \dfrac{a'}{b'} = \dfrac{a''}{b''} = $ etc.

with
$$a > a' > a'' > a''' > \cdots \geq 1$$
$$b > b' > b'' > b''' > \cdots \geq 1$$

But this is absurd.
(reductio ad absurdum) $\blacksquare$

So $\sqrt{2}$ is not a "number". But it's a perfectly good "length".



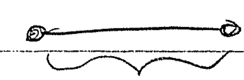
Greeks replaced:

"number" $\leftarrow$ "length of line segment".

This persisted until modern times.

Greek math based on ... Ruler & Compass.

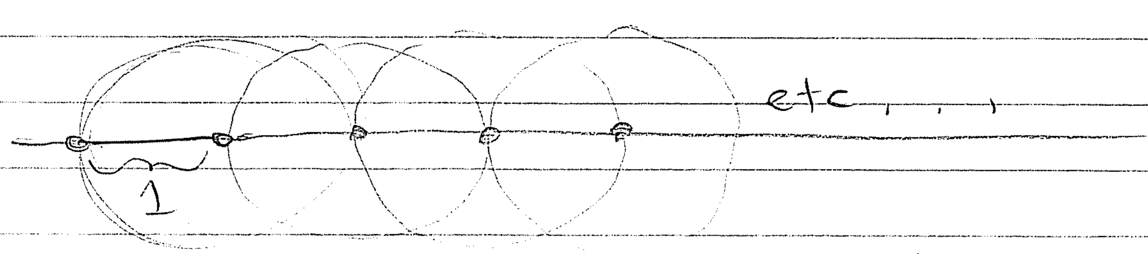
i.e. <u>lines</u> and <u>circles</u>.

Start from 

unit length = "1".

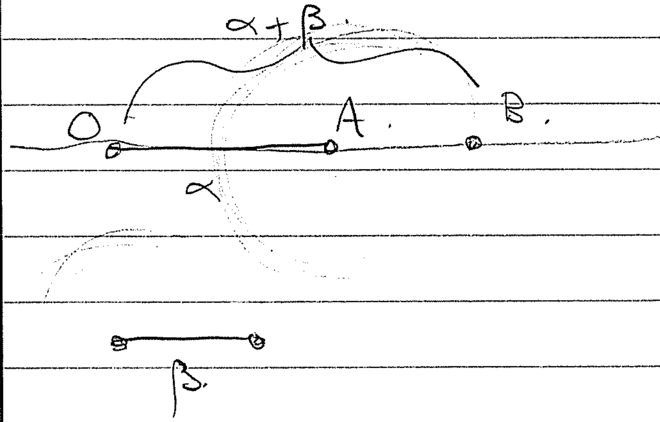
Which lengths can we construct?

(i.e. which "numbers" "exist")

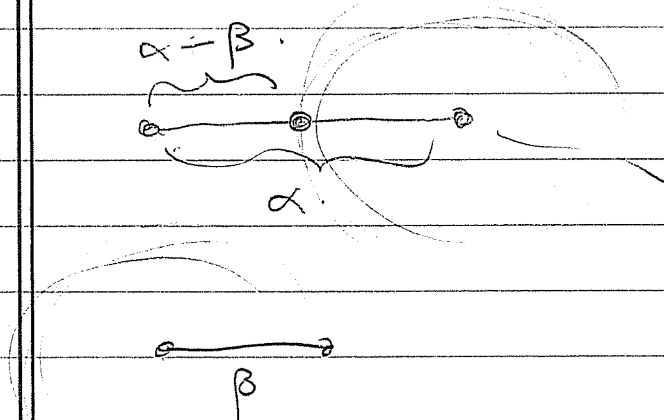


1, 2, 3, 4, ... all positive $\mathbb{Z}$.

Given α, β we can <u>add</u>: form radius β circle at A. to get $OB = \alpha + \beta$.



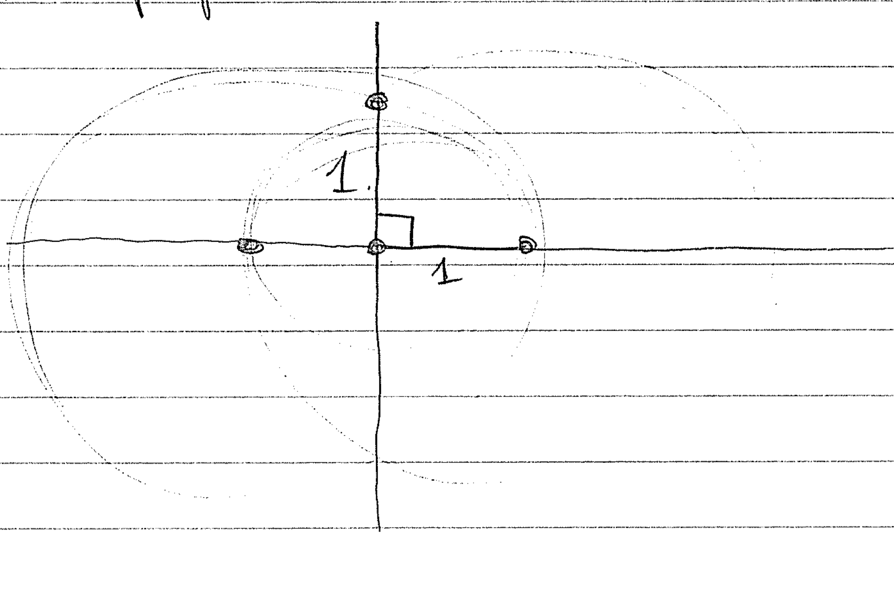
Given α, β we can <u>subtract</u>: same idea.



radius β.

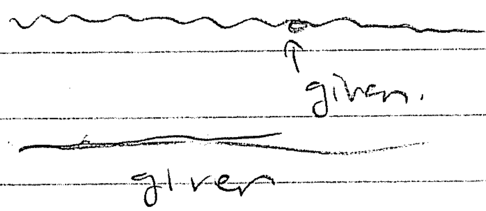
Given α, β we can <u>multiply</u>:

(1) we can form perpendicular axes.



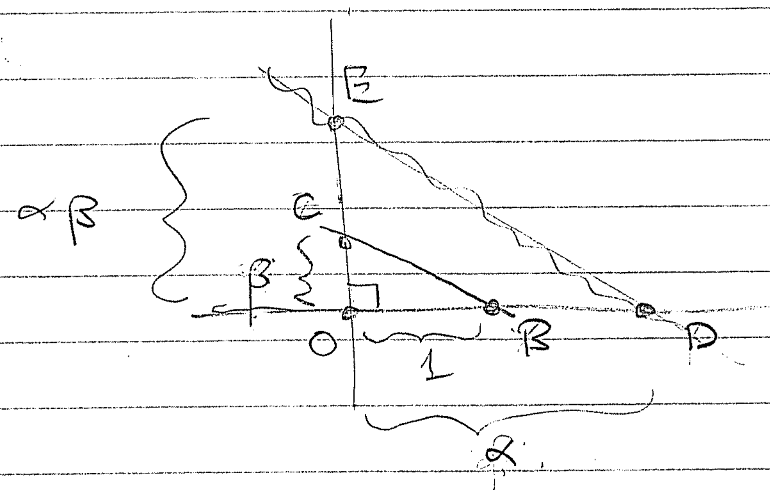
✓

② we can draw parallel to a given line



Proof (Euclid I.31).

③ Given:

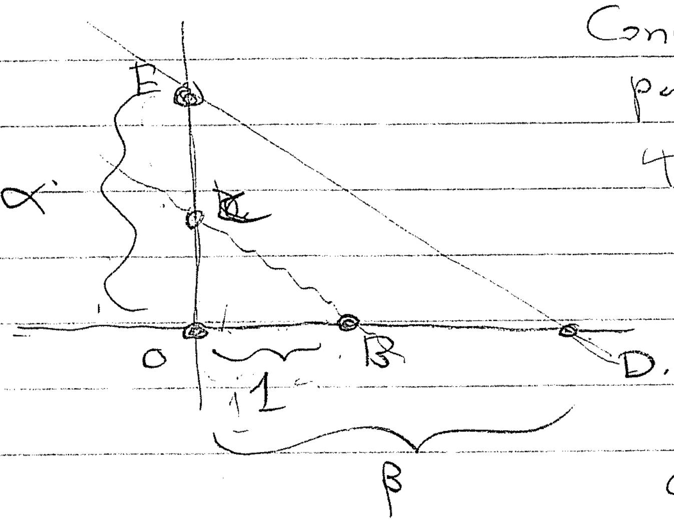


Draw DE parallel to BC.

Then $\dfrac{OD}{OB} = \dfrac{OE}{OC} \implies \dfrac{\alpha}{1} = \dfrac{OE}{\beta} \implies OE = \alpha\beta$.

Given α, β we can <u>divide</u>: same idea.

Given:



Construct BC parallel to DE.

Then $\dfrac{OD}{OB} = \dfrac{OE}{OC}$.

$\dfrac{\beta}{1} = \dfrac{\alpha}{OC}$.

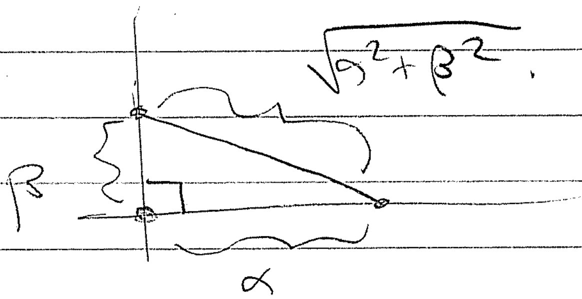
Get $OC = \dfrac{\alpha}{\beta}$.

✓

Conclusion: all positive rationals $\mathbb{Q}^+$ are constructible.

Is that all? NO!

Given α, β we can form $\sqrt{\alpha^2+\beta^2}$:



Pythagorean Theorem.

eg. $\sqrt{2} = \sqrt{1^2+1^2}$ is constructible.

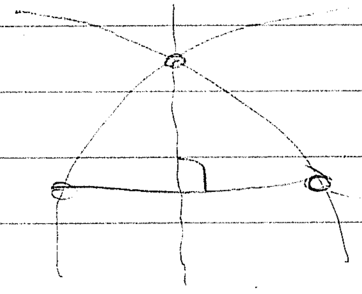
Is $\sqrt{3}$ constructible?

$3 \neq \alpha^2+\beta^2 \to$ no help.

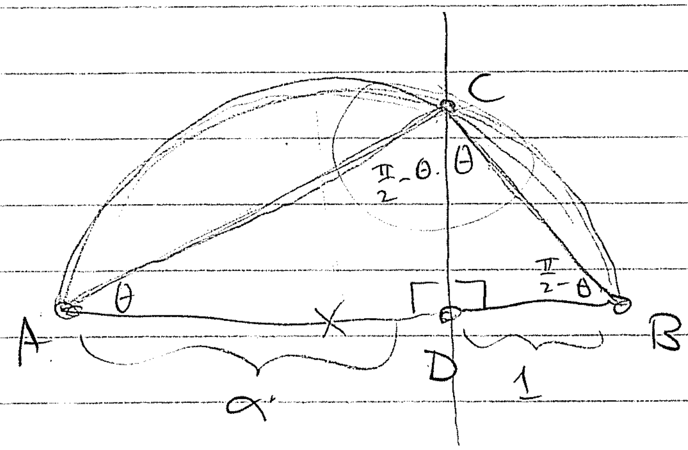
Theorem: If α is constructible, then so is $\sqrt{\alpha}$.

Proof: ① we can bisect a segment

(Euclid I.10)



(2) Given



Draw DC $\perp$ to AB.

Exercise: Show ACB is 90°.

Hence

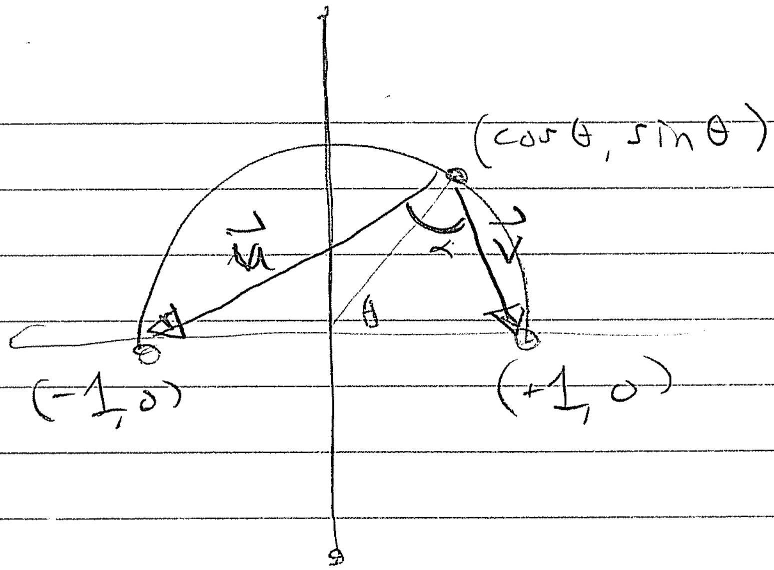
ΔABC, ΔADC, ΔCDB

are similar.

We get

$$\frac{AD}{CD} = \frac{CD}{BD} \Rightarrow \frac{\alpha}{CD} = \frac{CD}{1}.$$

next page.



point on the unit circle.

Consider vectors $\vec{u} = (\cos\theta + 1, \sin\theta)$
$\vec{v} = (\cos\theta - 1, \sin\theta)$.

What is the angle? between $\vec{u}, \vec{v}$,

Recall:

$$\vec{u} \circ \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\alpha.$$

$$\vec{u} \circ \vec{v} = 0 \iff \vec{u} \perp \vec{v}.$$

$$\vec{u} \circ \vec{v} = (\cos\theta - 1)(\cos\theta + 1) + \sin\theta \sin\theta.$$

$$= \cos^2\theta - 1^2 + \sin^2\theta$$

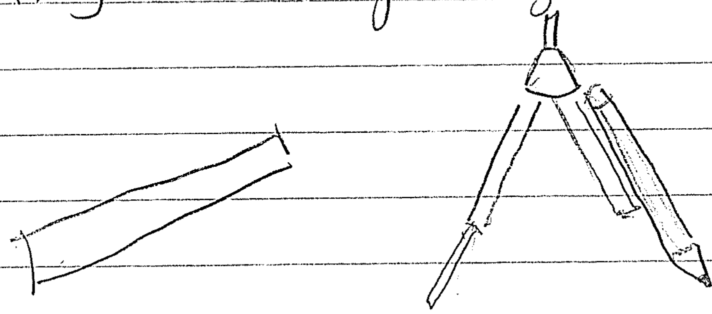
$$= \underbrace{\cos^2\theta + \sin^2\theta}_{1} - 1 = 0.$$

$\square$.

HW 2 due next Fri Mar 4

Today: Constructibility

Using a straightedge & compass



which "lengths" = "numbers" are constructible?

Start with an arbitrary unit length "1"

Last time we proved:

If α, β are constructible, then so are

(1) $\alpha + \beta$

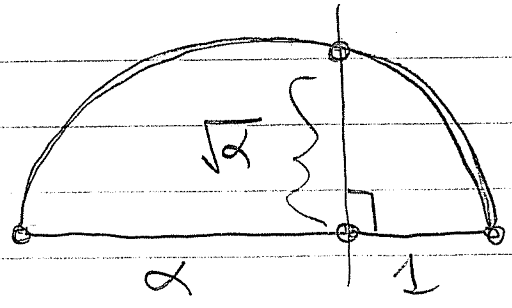
(2) $\alpha - \beta$ (when $\beta < \alpha$)

(3) $\alpha \cdot \beta$

(4) α / β

— All rational numbers are constructible.

(5) $\sqrt{\alpha}$.



So eg.

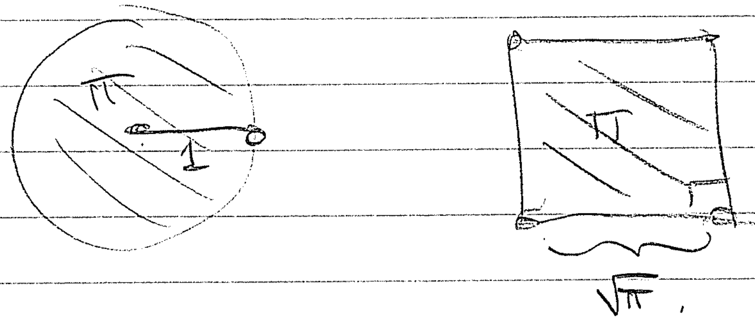
$$\sqrt{\frac{\sqrt{1+\sqrt{3}}}{5+\sqrt{2}} + \frac{101}{77}}$$

is constructible.

Q: Is *every* $\alpha \in \mathbb{R}$, $\alpha > 0$ constructible?
???
i.e. are "constructible lengths" = "all possible lengths"?
The Greeks didn't know, but they got stuck on 3 problems....

(1) Squaring the circle.
Given a circle, construct a square with the same area.

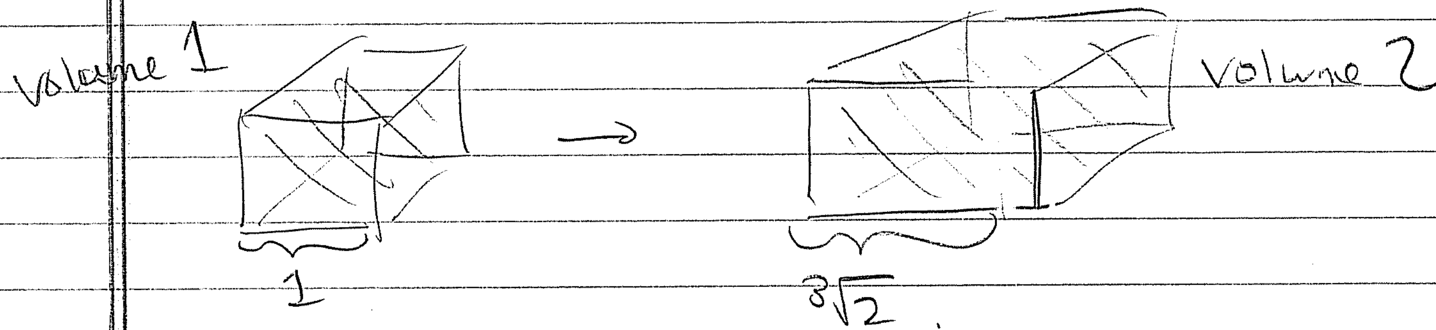
Unit circle has area π



Is $\sqrt{\pi}$ (or π) constructible?
Theorem (Lindemann, 1882): NO.

(2) Doubling the Cube.

Give (the edge of) a cube, construct (the edge of) a cube with double the volume.



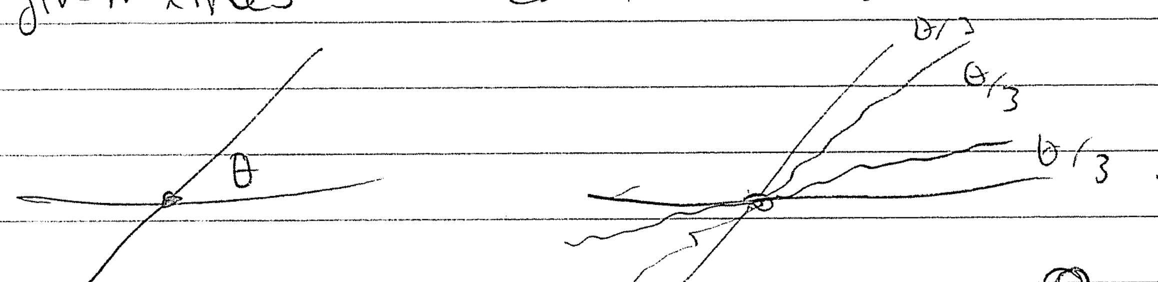
Is $\sqrt[3]{2}$ constructible?

Theorem (Descartes, 1637): NO.

(3) Trisecting an Angle

Given the angle θ, construct the angle $\frac{\theta}{3}$.

given lines construct lines



Given $\cos\theta$, is $\cos\left(\frac{\theta}{3}\right)$ always constructible?

Theorem (Gauss, 1796 and Wantzel, 1837): NO!

Q: How'd they do dat?

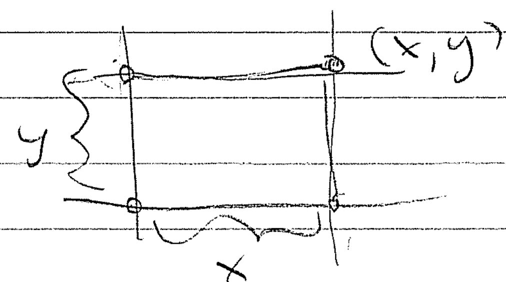
A: Algebra!

Let C = constructible numbers
D = numbers formed from
$1, +, -, \times, \div, \sqrt{\ }$.

Theorem: $C = D$.

Proof: We already saw $D \subseteq C$.
Need $C \subseteq D$.

Think in coordinates (Descartes). Note:
point (x,y) constructible $\Leftrightarrow$ x, y are cble.



So sp. (x,y) has been constructed. Want to show $x, y \in D$.

Where did (x,y) come from? It was on intersection point for some
line & line
line & circle
circle & circle

with cble coefficients. Claim: Then

$$x = \frac{a+\sqrt{b}}{c} \quad \text{and} \quad y = \frac{c+\sqrt{d}}{e}$$

for some constructible a, b, c, d, e, f.
Assume for induction that $a, b, c, d, e, f \in D$.
Then $x, y \in D$. $\blacksquare$

Intersect line & line $\rightarrow$ linear equation
line & circle $\rightarrow$ quadratic
circle & circle $\rightarrow$?

you will do
this on HW 3

HW 3 due Friday.

Office Hours

Wed & Thurs 2:30–4:00.

Today: $F[\sqrt{c}]$.

Recall: If $\alpha \in \mathbb{R}$ is constructible (with straight edge and compass) then α is formed recursively by intersecting lines and circles with cble coefficients.

In each case the solution can be done using $+, -, \times, \div, \sqrt{\ }$

① line $\cap$ line $\rightsquigarrow$ linear equation

② line $\cap$ circle $\rightsquigarrow$ quadratic equation

③ circle $\cap$ circle $\rightsquigarrow$? Exercise.

Hence α has an expression in $1, +, -, \times, \div, \sqrt{\ }$

"a degree 2 algebraic expression"

Conversely, we have seen that $\alpha \pm \beta$, $\alpha\beta$, $\frac{\alpha}{\beta}$, $\sqrt{\alpha}$ can be constructed, so any $\alpha \in \mathbb{R}$ with a deg 2 alg. expression is constructible.

Summary: Let C = constructible #'s
D = #'s formed recursively from $1, +, -, \times, \div, \sqrt{\ }$.

Then $C = D$.
(geometry) (Algebra!)

(A means to show that some $\alpha \in \mathbb{R}$ is NOT c'ble.)

Describe D more precisely?

Let F = a field
"number system with $+, -, \times, \div$"
(eg. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$).
and suppose $c \in F$ with $\sqrt{c} \notin F$.
We form a new number system

$F[\sqrt{c}]$ = "F adjoin $\sqrt{c}$"

$$= \{a + b\sqrt{c} : a, b \in F\}.$$

eg. $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$
"rational numbers adjoin $\sqrt{2}$".

eg. $\mathbb{R}[\sqrt{-1}] = \{a + b\sqrt{-1} : a, b \in \mathbb{R}\}$

In general $F[\sqrt{c}]$ is similar to $\mathbb{R}[\sqrt{-1}]$.

We can divide:

eg. Given $a + b\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$

$$\frac{1}{a+b\sqrt{2}} = \frac{1}{(a+b\sqrt{2})} \frac{a-b\sqrt{2}}{(a-b\sqrt{2})} = \frac{a-b\sqrt{2}}{a^2-2b^2}.$$

$$= \left(\frac{a}{a^2-2b^2}\right) + \left(\frac{-b}{a^2-2b^2}\right)\sqrt{2} \in \mathbb{Q}[\sqrt{2}].$$

$\Longrightarrow$ $\mathbb{Q}[\sqrt{2}]$ is a field.

The map $\begin{array}{ccc} \mathbb{Q}[\sqrt{2}] & \longrightarrow & \mathbb{Q}[\sqrt{2}] \\ a + b\sqrt{2} & \longmapsto & a - b\sqrt{2} \end{array}$
is called CONJUGATION.

$$v \longmapsto \bar{v}$$

$\mathbb{Q}[\sqrt{2}]$ is also a vector space.

suppose $a + b\sqrt{2} = c + d\sqrt{2}$

Then $(a-c) = (d-b)\sqrt{2}$.

If $b \neq d$, then $\sqrt{2} = \frac{a-c}{d-b} \in \mathbb{Q}$ a contradiction!

Hence $b = d$ & $a = c$

Summary:

$$a + b\sqrt{2} = c + d\sqrt{2} \iff a = c \ \& \ b = d.$$

So $a + b\sqrt{2}$ acts like a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ (geometry)

$$\mathbb{Q}[\sqrt{2}] \approx \mathbb{Q}^2$$

the rational plane.

Summary: Given field F and $c \in F$ with $\sqrt{c} \notin F$, then

$F[\sqrt{c}]$ is a field. with

$$a + b\sqrt{c} = a' + b'\sqrt{c} \iff a = a' \text{ AND } b = b'$$

$F \subseteq F[\sqrt{c}]$

is a "quadratic field extension"

So WHAT?

Rephrase constructibility:

$\alpha \in \mathbb{R}$ is constructible

$\Updownarrow$

$\exists$ chain of quadratic extensions

$$\mathbb{Q} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_r \subseteq \cdots\cdots \subseteq \mathbb{R}.$$

with $\alpha \in F_r$.

This is USEFUL

Theorem: $\sqrt[3]{2}$ is not constructible.

(Landau, when he was a student).

Proof: Suppose (for contradiction) that $\sqrt[3]{2}$ is constructible. Then $\exists$.

$$\mathbb{Q} \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k \subseteq F_{k+1} \cdots \subseteq \mathbb{R}$$

where $\sqrt[3]{2} \in F_{k+1}$ but not $\in F_k$.

Hence $\sqrt[3]{2} = a + b\sqrt{c}$ with $a, b, c \in F_k$
$\sqrt{c} \notin F_k$.

CUBE to get

$$2 + 0\sqrt{c} = (a + b\sqrt{c})^3$$
$$= (a^3 + 3ab^2c) + (3a^2b + b^3c)\sqrt{c}.$$

compare coefficents:

(*) $2 = a^3 + 3ab^2c$ & $0 = 3a^2b + b^3c$

Now (for fun) expand

$$(\boxed{a - b\sqrt{c}})^3 - 2$$

$$= (a^3 + 3ab^2c - 2) - (3a^2b + b^3c)\sqrt{c}.$$

from(*) $\overset{\text{from}(*)}{=} 0 - 0\sqrt{c} = 0.$

Hence. $a - b\sqrt{c}$ is a Real cube root of 2.

Conclude $\sqrt[3]{2} = a + b\sqrt{c} = a - b\sqrt{c}$.

$\Longrightarrow a = a$ & $b = -b \Longrightarrow b = 0$

$\Longrightarrow \sqrt[3]{2} = a \in F_k$ contradiction $\blacksquare$

~~Cleaner~~

Cleaner Proof:

Suppose $\sqrt[3]{2}$ is constructible. Then $\exists$

$$\mathbb{Q} \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k \subseteq F_{k+1} \subseteq \cdots \subseteq \mathbb{R}$$

$$\sqrt{c_1} \quad \sqrt{c_2} \quad \sqrt{c_3} \qquad \sqrt{c_k} \quad \sqrt[3]{2}.$$

where $\sqrt[3]{2} \in F_{k+1}$ but NOT $\in F_k$.

Then we can write $\sqrt[3]{2} = a + b\sqrt{c_k}$, $a, b \in F_k$.

So $(a + b\sqrt{c_k})^3 - 2 = 0$.

Apply CONJUGATION $F_{k+1} \longrightarrow F_{k+1}$.

$$\alpha + \beta\sqrt{c_k} \longmapsto \alpha - \beta\sqrt{c_k}$$

$$\overline{(a + b\sqrt{c_k})^3 - 2} = \overline{0}$$

automorphism

$$(\overline{a + b\sqrt{c_k}})^3 - 2 = 0$$

$$(a - b\sqrt{c_k})^3 - 2 = 0$$

$\Longrightarrow a - b\sqrt{c_k} \in \mathbb{R}$ is a cube root of 2. $\lightning$

HW due Friday

O.H

Today & Tomorrow

2:30 – 4:00

MATH CLUB

TODAY

5PM

Ungar 402

Note: DO NOT hand in Problem A.5.

<u>I'll do it.</u>

Today: A "high-school" problem

Compute intersection of circles. (Easy?)

$$\begin{cases} (x-a)^2 + (y-b)^2 = R^2 \\ (x-c)^2 + (y-d)^2 = r^2 \end{cases}$$

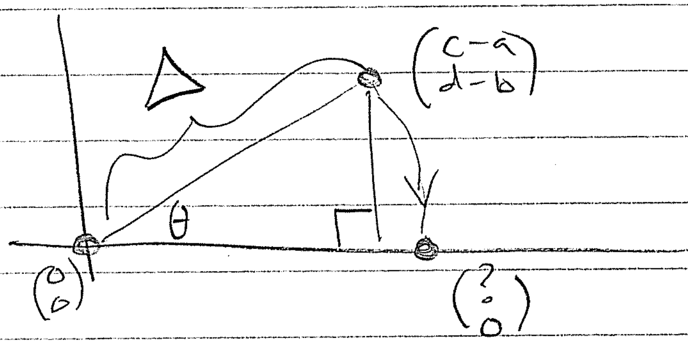
Try <u>to simplify</u> by changing coordinates.

Idea: move centers (a,b) and (c,d) to $(0,0)$ and $(?,0)$.

First let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be <u>translation</u> by $\begin{pmatrix} -a \\ -b \end{pmatrix}$

$$T\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

So $T\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ & $T\begin{pmatrix}c\\d\end{pmatrix} = \begin{pmatrix}c-a\\d-b\end{pmatrix}$



Let $\Delta = \sqrt{(c-a)^2 + (d-b)^2} = \text{dist}\left(\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}c\\d\end{pmatrix}\right)$

Next <u>Rotate</u> by $-\theta$

$R_{-\theta}: \mathbb{R}^2 \to \mathbb{R}^2.$

$$R_{-\theta} = \begin{pmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

$$= \frac{1}{\Delta}\begin{pmatrix}c-a & d-b\\ -(d-b) & c-a\end{pmatrix}.$$

Does this work?

$R_{-\theta}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ &.

$$R_{-\theta}\begin{pmatrix}c-a\\d-b\end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix}c-a & d-b\\ -(d-b) & c-a\end{pmatrix}\begin{pmatrix}c-a\\d-b\end{pmatrix}$$

$$= \frac{1}{\Delta} \begin{pmatrix} (c-a)^2 + (d-b)^2 \\ -(c-a)(d-b) + (c-a)(d-b) \end{pmatrix}$$

$$= \frac{1}{\Delta} \begin{pmatrix} \Delta^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta \\ 0 \end{pmatrix}.$$

The transformation is

$$\begin{pmatrix} x \\ y \end{pmatrix} \to R\left(T\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$= R \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

$$= \frac{1}{\Delta} \begin{pmatrix} c-a & d-b \\ -(d-b) & c-a \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

$$= \frac{1}{\Delta} \begin{pmatrix} (x-a)(c-a) + (y-b)(d-b) \\ -(x-a)(b-b) + (y-b)(c-a) \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

new coordinates.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = RT\begin{pmatrix} x \\ y \end{pmatrix}$$

How to invert?

The inverse of RT is $T^{-1}R^{-1}$

where T^{-1} = translate by $(+a, +b)$
R^{-1} = rotate by $+\theta$.

Formula on handout.

Note: If $\begin{pmatrix} x \\ y \end{pmatrix}$ is on circles

(a,b) radius R
(c,d) radius r

Then $\begin{pmatrix} x' \\ y' \end{pmatrix} = RT\begin{pmatrix} x \\ y \end{pmatrix}$ is on circles

$(0,0)$ radius R
$(\Delta, 0)$ radius r

Hence

$$\left.\begin{aligned} x'^2 + y'^2 &= R^2 \\ (x'-\Delta)^2 + y'^2 &= r^2. \end{aligned}\right\}$$

Much easier to solve $\rightsquigarrow$ A.6.

Recall:

Given field F with $c \in F$, $\sqrt{c} \notin F$,
define Quadratic Field Extension

$$F \subseteq F[\sqrt{c}] = \{ a + b\sqrt{c} \;:\; a, b \in F \}.$$

similar to Complex numbers.

$$\mathbb{R} \subseteq \mathbb{R}[\sqrt{-1}].$$

We say $\alpha \in \mathbb{R}$ has a "degree 2 alg. expression" if α can be written from $1, +, -, \times, \div, \sqrt{\ }$.

e.g. $\alpha = \sqrt{1 + \sqrt{3}}$

Equivalently, $\exists$ chain of Q.F.E.

$$\mathbb{Q} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq \mathbb{R}$$

s.t. $\alpha \in F_i$ for some i.

e.g.

chain

$$\mathbb{Q} \subseteq \mathbb{Q}[\sqrt{3}] \subseteq \mathbb{Q}[\sqrt{3}][\sqrt{1+\sqrt{3}}]$$

3 ✓ $\qquad \ni 1+\sqrt{3}$ $\qquad \ni \sqrt{1+\sqrt{3}}$.

$\sqrt{3}$ ✗ $\qquad \cancel{\sqrt{1+\sqrt{3}}}$

$\longrightarrow$ more "nesting".

Theorem: $\sqrt[3]{2}$ is not constructible.

Proof: suppose (for contradiction) that it **is**. Then $\exists$

$$\mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq \mathbb{R}.$$

where $\sqrt[3]{2} \in F_{k+1}$ for some k.
$\sqrt[3]{2} \notin F_k$

(Note: $\sqrt[3]{2} \notin F_0 = \mathbb{Q}$).

Say. $F_{k+1} = F_k[\sqrt{c_k}] \cancel{= \{a+b\sqrt{c_k} : a, b \in F_k\}}$

Hence $\sqrt[3]{2} = a + b\sqrt{c_k}$ for some $a, b \in F_k$.
(Note: $b \neq 0$ because $\sqrt[3]{2} \notin F_k$).

Then $(a+b\sqrt{c_k})^3 = 2$.
CONJUGATE BOTH SIDES

$$\overline{(a+b\sqrt{c_k})^3} = \overline{2 + 0\sqrt{c_k}}.$$

$$(\overline{a+b\sqrt{c_k}})^3 = 2 - 0\sqrt{c_k}.$$

$$(a-b\sqrt{c_k})^3 = 2$$

Hence $a - b\sqrt{c_k} \in \mathbb{R}$ is a cube root 2.
Hence $a - b\sqrt{c_k} = a + b\sqrt{c_k} \Rightarrow b = -b \Rightarrow b = 0$ ↯ $\square$

→ HW 3 due now.

→ HW 4 due next Friday March 11.

→ Spring Break

→ Exam 2, Fri Mar 25

Today: $\cos\theta \overset{?}{\rightsquigarrow} \cos\frac{\theta}{3}$

We have seen

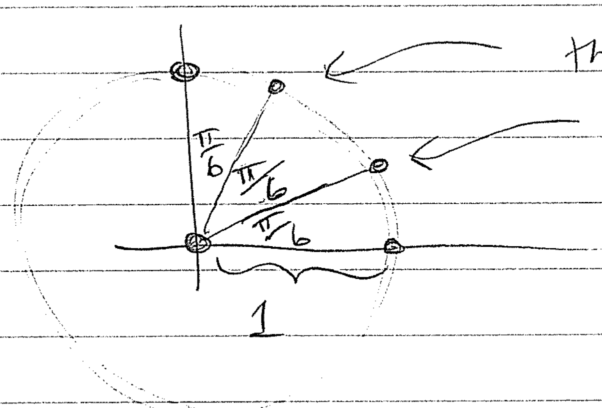
$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}.$$

Hence if $\cos\theta$ is c'ble then $\cos\frac{\theta}{2}$ is c'ble

"any c'ble angle can be bisected"

Some angles can be trisected.

eg. a right angle

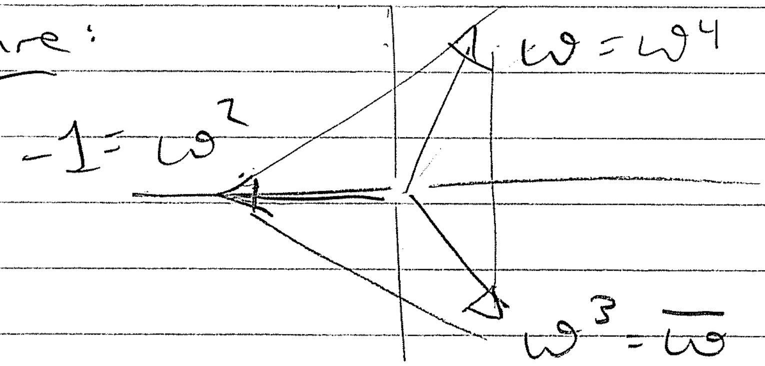


these points are c'ble.

$$60^\circ \qquad 60^\circ$$

Proof: Let $\omega = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

Picture:



cube roots of -1.

$$\omega + \omega^2 + \omega^3 = 0$$
$$\omega - 1 + \overline{\omega} = 0$$
$$\omega + \overline{\omega} = 1$$
$$2\cos\left(\frac{\pi}{3}\right) = 1$$
$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \Rightarrow \sin\left(\frac{\pi}{3}\right) = \sqrt{1 - \frac{1}{2}}$$
$$= \frac{\sqrt{3}}{2}.$$

Both c'ble!

$\boxtimes$

However,

Claim: $\frac{\pi}{3}$ cannot be trisected.

i.e. $\cos\left(\frac{\pi}{9}\right)$ is not c'ble.

Let's prove it!

Recall: $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.

Put $\theta = \pi/9$ to get

$$\frac{1}{2} = \cos\left(\frac{\pi}{3}\right) = 4\cos^3\left(\frac{\pi}{9}\right) - 3\cos\left(\frac{\pi}{9}\right).$$

$$4x^3 - 3x - \frac{1}{2} = 0, \quad \text{where } x = \cos\left(\frac{\pi}{9}\right).$$

Let $y = x/2$ to get

$$\frac{4y^3}{8} - \frac{3y}{2} - \frac{1}{2} = 0$$

$$y^3 - 3y - 1 = 0 \quad \text{where } y = \frac{\cos\left(\frac{\pi}{9}\right)}{2}.$$

<u>Claim</u>: $y^3 - 3y - 1 = 0$ has <u>no solution</u> in $\mathbb{Q}$

<u>Proof</u>: Suppose $y = \frac{a}{b}$ is a solution with $a, b \in \mathbb{Z}$, a, b coprime (no common factor)

Then $\frac{a^3}{b^3} - \frac{3a}{b} - 1 = 0$

$$a^3 - 3ab^2 - b^3 = 0.$$

$$a(a^2 - 3b^2) = b^3$$

$\Rightarrow a \mid b^3$ But a, b coprime.

Hence $a = \pm 1$

Similarly $a^3 = b(3a + b^2)$

$\Rightarrow b \mid a^3 \Rightarrow b = \pm 1$.

So the only possible $\mathbb{Q}$ roots are $\frac{\pm 1}{\pm 1} = \pm 1$.

But

$$(+1)^3 - 3(+1) - 1 \neq 0$$
$$(-1)^3 - 3(-1) - 1 \neq 0$$

Corollary: $\cos\left(\frac{\pi}{9}\right) \notin \mathbb{Q}$

General method.

Rational Root Test:

Given $f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \in \mathbb{Z}[x]$.

If $f\left(\frac{a}{b}\right) = 0$ for $\frac{a}{b} \in \mathbb{Q}$. Then

$$a \mid c_0 \quad \text{AND} \quad b \mid c_n.$$

$\Rightarrow$ Finitely many possibilities that can be checked.

eg. $3x^3 - 5x^2 + 5x - 2 = 0$.

$\mathbb{Q}$-roots restricted to $\frac{\pm 1, 2}{\pm 1, 3} = \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$

plug them in.

Lemma: Consider Q.F.E. $F \subseteq F[\sqrt{c}]$ with conjugation $a+b\sqrt{c} \mapsto a-b\sqrt{c}$.
If cubic $f(x) \in F[x]$ has a root in $F[\sqrt{c}]$, then it also has a root in F.

Proof: Suppose $\alpha \in F[\sqrt{c}]$ with $f(\alpha) = 0$.
If $\alpha \in F$ done. Otherwise note that

$$\overline{f(\alpha)} = \overline{0}$$

$$f(\overline{\alpha}) = 0$$

So $\overline{\alpha}$ is another root ($\alpha \neq \overline{\alpha}$ since $\alpha \notin F$)
Factor to get

$$f(x) = (x-\alpha)(x-\overline{\alpha})(x-\beta) \text{ for some } \beta.$$
$$= (x^2 - (\alpha+\overline{\alpha})x + \alpha\overline{\alpha})(x-\beta)$$

But note $\alpha + \overline{\alpha} \in F$
$\alpha\overline{\alpha} \in F$.

Hence $\beta = x - \underbrace{\frac{f(x)}{(x^2-(\alpha+\overline{\alpha})x+\alpha\overline{\alpha})}}_{\text{all in } F[x]} \in F$

$\blacksquare$

Theorem: $\cos\left(\frac{\pi}{9}\right)$ is NOT constructible.

Proof: Suppose (for contradiction) that it IS c'ble.

Then $\exists$ chain of QFE:

$$\mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k \subseteq \cdots \subseteq \mathbb{R}$$

such that $\cos\left(\frac{\pi}{9}\right) \in F_k$.

But then $x^3 - 3x - 1$ has a root in F_k.
Hence it has a root in F_{k-1} (by Lemma)
- - - - - - in F_{k-2}.

Hence it has a root in $F_0 = \mathbb{Q}$.

But $x^3 - 3x - 1$ has no $\mathbb{Q}$-root $\blacksquare$

Trisecting an Angle is Impossible!
(in general)