HW 2 due Fri Feb 11
- post tomorrow
Exam 1: Fri Feb 18

Math Club Today 5PM.

Today: $\mathbb{C}$.

Definition 1

$$\mathbb{C} := \{ a+bi : a, b \in \mathbb{R} \}$$

where $i^2 = -1$ (but what is i?)

sum $(a+bi)+(c+di) = (a+c)+(b+d)i$

product $(a+bi)(c+di) = ac + bci + adi + bdi^2$
$= (ac-bd) + (ad+bc)i$

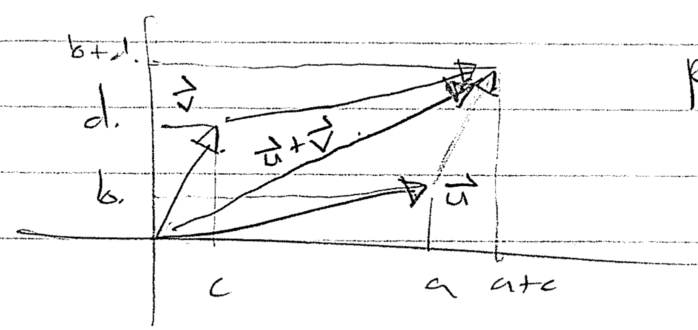
Definition 2

$\mathbb{C} := \mathbb{R}^2$ the real plane
with special $+$, $\times$.

vector addition.

sum $(a,b)+(c,d) = (a+c, b+d)$

picture.



parallelogram law.

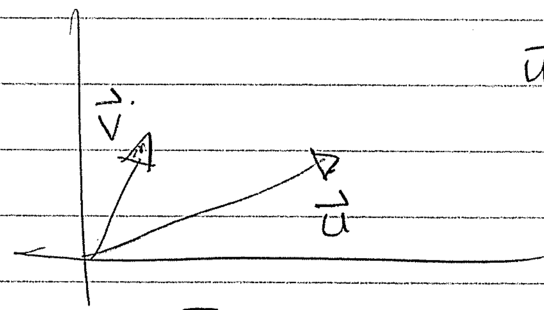
How to "multiply" vectors?

$$\underbrace{(a,b)(c,d)}_{\text{juxtaposition}} = (ac-bd,\ ad+bc)$$

"complex" multiplication

Picture?

$\vec{u}\vec{v} = ?$



IOU.

$$\text{Note: } \overset{\text{Geometry.}}{(a,b) \in \mathbb{R}^2} \overset{\text{isomorphism.}}{\Longleftrightarrow} \overset{\text{Algebra.}}{a+ib \in \mathbb{C}}.$$

$$(1,0) \longleftrightarrow 1$$

~~The Key Insight~~. $(0,1) \Longleftrightarrow i$

The Key Insight: Think of each (a,b) as a function $\mathbb{R}^2 \to \mathbb{R}^2$ by

$$(a,b)(x,y) = (ax-by,\ ay+bx).$$

$$= \underbrace{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{matrix multiplication.}}.$$

What does $1_{\mathbb{C}} = (1,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ do?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

<u>the identity map.</u>

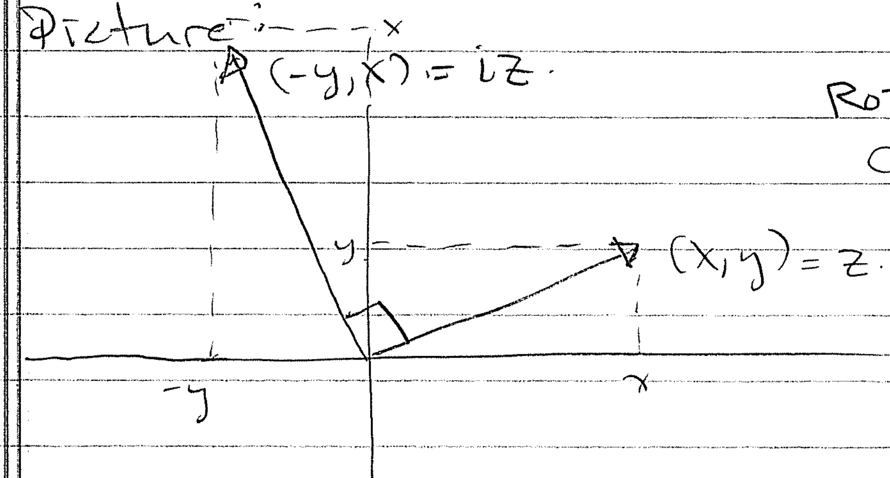
Makes sense: $\underset{\text{identity.}}{1}z = z \quad \forall z \in \mathbb{C}.$

What does $i_{\mathbb{C}} = (0,1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ do?

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

" $i(x+iy) = -y + ix$ ".

Picture:



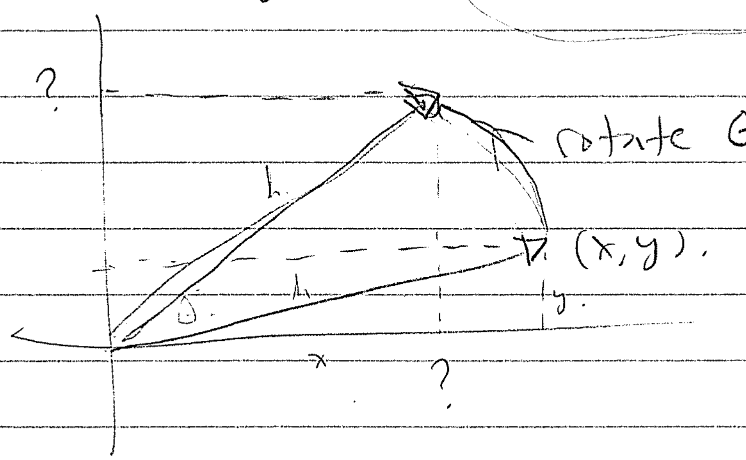
ROTATE $90°$
Counterclockwise.

1 = identity.
i = rotate $90°$.

How to rotate by θ ?

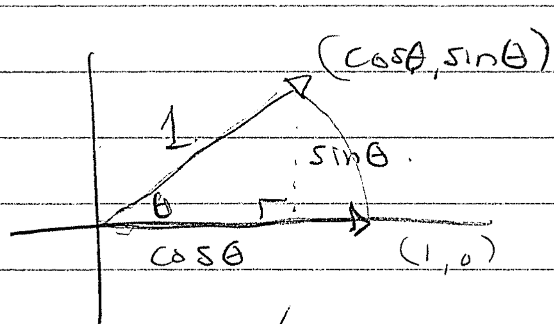
$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2.$$

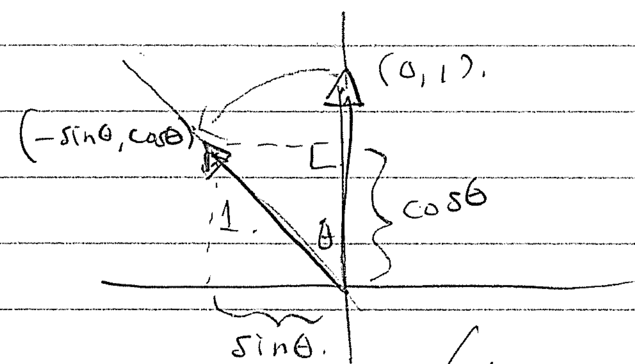
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$



$$\cos\theta = \frac{x}{h}.$$

Proof: First rotate $(1,0)$ and $(0,1)$.





$$R_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad R_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Finally;

linear!

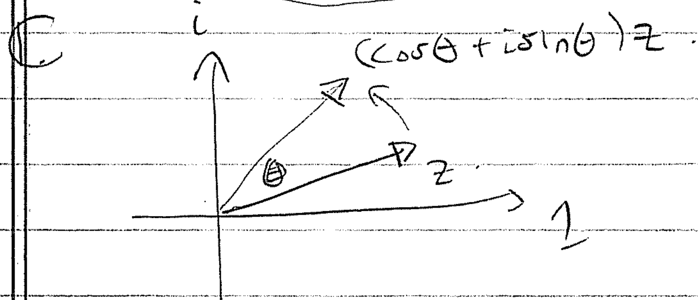
$$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = R_\theta \left(x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = x R_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y R_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= x \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + y \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \blacksquare$$

$\text{cis}\,\theta := \cos\theta + i\sin\theta.$

Conclusion: in $\mathbb{C}$,

$$(\cos\theta + i\sin\theta)\,z = z \text{ rotated by } \theta.$$



De Moivre's Theorem (1730)

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)}$$

$(\text{cis}\,\theta)^n = \text{cis}(n\theta).$

Proof: Let $z \in \mathbb{C}$ be any complex number $\neq 0$.

One one hand: $\text{cis}(n\theta)\,z = z$ rotated by $n\theta$.

On other hand:

$$\underbrace{\text{cis}\,\theta \cdots \text{cis}\,\theta\;\text{cis}\,\theta}_{n \text{ times}} \cdot z = (\text{cis}\,\theta)^n z = z \text{ rotated by } n\theta.$$

Hence $\text{cis}(n\theta)\,z = (\text{cis}\,\theta)^n z.$

Since $z \neq 0$, divide to get

$$(\text{cis}\,\theta)^n = \text{cis}(n\theta). \quad \blacksquare$$

Corollary: Put $\alpha = \frac{\theta}{n}$ to get

$$\left(\cos\frac{\alpha}{n} + i\sin\frac{\alpha}{n}\right)^n = \cos\alpha + i\sin\alpha.$$

an nth root of $\cos\alpha + i\sin\alpha$.

//

Application: Solve $x^3 - 1 = 0$.

$$x^3 = 1$$

Old way: $x^3 - 1 = (x-1)(x^2+x+1)$

$$x = 1 \quad \text{OR} \quad x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

New Way: which numbers cube to 1?

$$1 = \cos(2\pi k) + i\sin(2\pi k) \text{ for any } k \in \mathbb{Z}.$$
$$= \operatorname{cis}(2\pi k).$$

Hence

$$\left(\operatorname{cis}\left(\frac{2\pi k}{3}\right)\right)^3 = \operatorname{cis}(2\pi k) = 1 \quad \forall k \in \mathbb{Z}.$$

$\operatorname{cis}\left(\frac{2\pi k}{3}\right)$ is cube root of 1 $\forall k$.

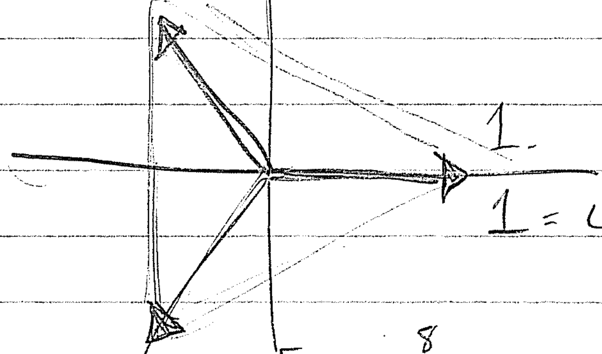
$$\| $$

$\left(\operatorname{cis}\left(\frac{2\pi}{3}\right)\right)^k$. rotate by $\frac{2\pi}{3} = 120°$ k times.

$\mathbb{C}$.

Picture. $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$\omega=\omega^4=\omega^7 \dots$



1.

$1=\omega^3=\omega^6 \dots$

$\omega^2=\omega^5=\omega^8 \dots$ $\qquad \omega^k = cis\left(\frac{2\pi k}{3}\right)$

$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Take $k=0,1,2$

$cis(0) = \underline{1}$

$cis(2\pi/3) = \cos\left(2\pi/3\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$cis(4\pi/3) = \cos(4\pi/3) + i\sin(4\pi/3) = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$.

I lost my voice

HW 1 Average 30/32.
HW 2 due Fri Feb 11
Exam 1 — Fri Feb 18

Today: Polar form of $\mathbb{C}$.

Recall the definition(s)

$\mathbb{C} = \{ a + ib : a, b \in \mathbb{R} \}$
with $i^2 = -1$

☆ But what *is* i?

$\mathbb{C} = \mathbb{R}^2 = \{ (a, b) : a, b \in \mathbb{R} \}$
with funny multiplication

☆ $(a, b)(c, d) := (ac - bd, ad + bc)$

THE BEST

$\mathbb{C} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$

with matrix $+, \times$ NATURAL :)

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{\text{identity} \\ \text{"1"}}} + b \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\substack{\text{rotate } 90^\circ \\ \text{"}i\text{"}}} \leftrightarrow a + ib.$$

THE KEY to $\mathbb{C}$

Recall: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$

rotate θ counterclockwise.

$$\Longrightarrow \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix}$$

rotate θ, n times

rotate $n\theta$, one time.

$\Longrightarrow$ De Moivre's formula (1730)

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)}$$

~~THE KEY~~

"$\operatorname{cis}^n(\theta) = \operatorname{cis}(n\theta)$."

Application

$$\cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2$$

$$= \cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta$$

$$= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$$

Conclusion:

$$\left.\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta \end{aligned}\right\} \text{double angle formulas.}$$

More generally:

$$\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = ?$$

rotate by α,
then rotate by β.

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

rotate by $\alpha+\beta$.

Hence

$$\boxed{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)}$$

Better than de Moivre.

"$\cancel{\mathrm{cis}(\alpha)\,\mathrm{cis}(\beta) = \mathrm{cis}(\alpha+\beta)}$"

Application:

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \cos\beta\sin\alpha)$$

$$\Longrightarrow \quad \begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \cos\alpha\sin\beta + \cos\beta\sin\alpha \end{aligned}$$

"angle sum formulas

Rephrase: $\boxed{\operatorname{cis}(\alpha+\beta) = \operatorname{cis}(\alpha)\operatorname{cis}(\beta)}$

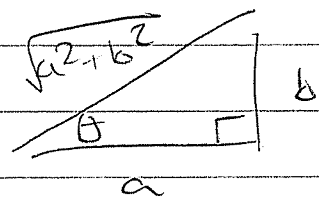
VERY interesting property.

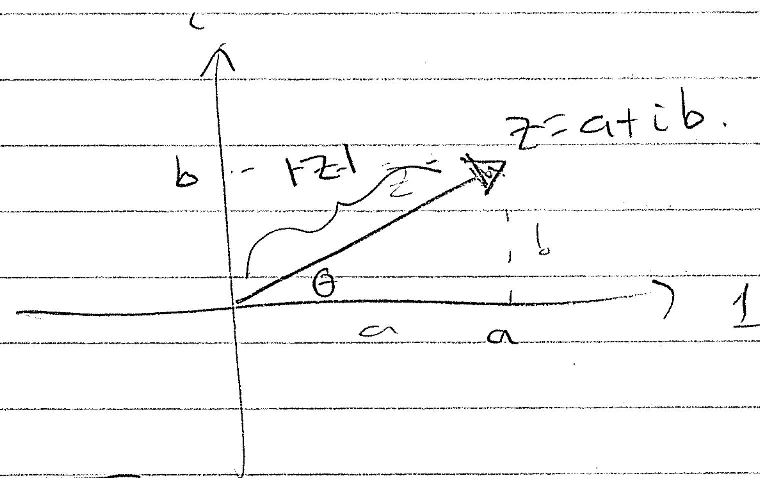
$\operatorname{cis}: \mathbb{R} \to \mathbb{C}$.

sends addition $\to$ multiplication.

Application: THE POLAR FORM.

Given $z \in \mathbb{C}$





Let $|z| = \sqrt{a^2+b^2}$. the "length" or "modulus" of z.

$\theta = $ (angle or "argument" of z) $= \tan^{-1}\left(\frac{b}{a}\right)$

Now watch:

$$z = (a+ib) = \sqrt{a^2+b^2}\left[\left(\frac{a}{\sqrt{a^2+b^2}}\right) + i\left(\frac{b}{\sqrt{a^2+b^2}}\right)\right]$$

$$= |z|\left[\cos\theta + i\sin\theta\right] = |z|\operatorname{cis}(\theta).$$

Summary: Every $z = a + ib \in \mathbb{C}$ can be written as -

$$a + ib = |z|(\cos\theta + i\sin\theta)$$
$$= |z| \operatorname{cis}(\theta)$$

The Polar Form

where $|z| = \sqrt{a^2+b^2} \geq 0$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Finally, we understand multiplication.

Given $z, w \in \mathbb{C}$, let

$$z = r \operatorname{cis}(\alpha) \qquad r, s, \alpha, \beta \in \mathbb{R}$$
$$w = s \operatorname{cis}(\beta) \qquad r, s \geq 0.$$

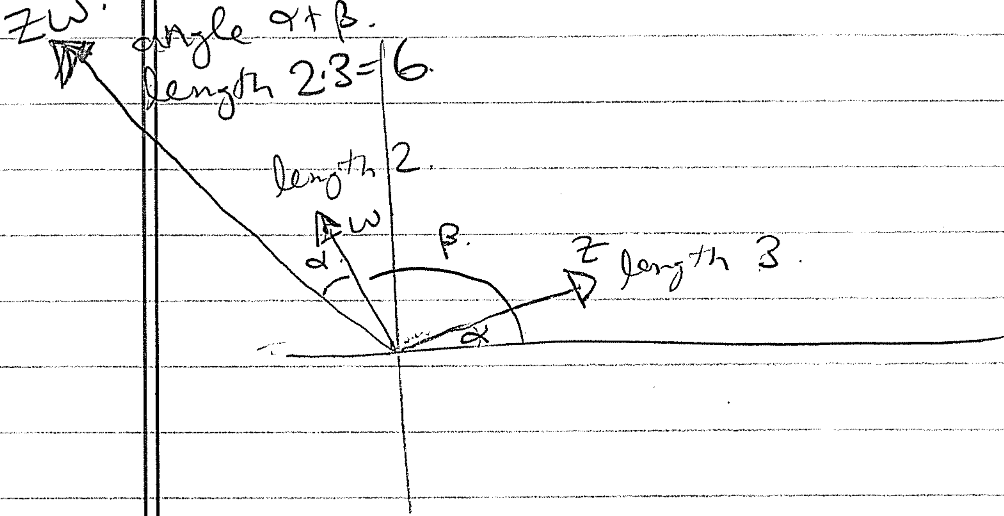
Then

$$zw = r \operatorname{cis}(\alpha)\, s \operatorname{cis}(\beta)$$
$$= rs \operatorname{cis}(\alpha) \operatorname{cis}(\beta)$$
$$= rs \operatorname{cis}(\alpha+\beta)$$

lengths multiply

angles add.

Picture eg. multiply



~~What does $cis(\alpha+\beta) = cis(\alpha)\, cis(\beta)$ REMIND you of?~~

The matrix analogue:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{a^2+b^2} \end{pmatrix} \begin{pmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}$$

dilation rotation.

pos. def. unitary.

$$M = P \cdot U.$$

Next: Roots of Unity
(or the Exponential Function)

HW 2 due Fri.
NOTE: TYPO
A.6(c) should read

"$|z|^2 = \det F(z)$"

Exam 1 Fri Feb 18

Today: Roots of Unity.

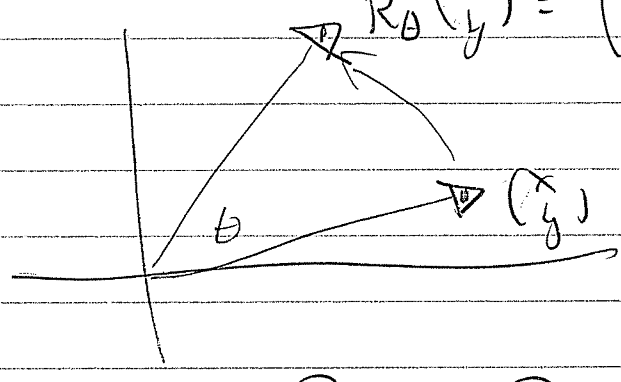
Recall: Landmarks.

Theorem: $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

is a rotation counterclockwise by θ.

Pic.

$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$



Corollary. $R_\alpha R_\beta = R_{\alpha+\beta}$

matrix mult.

Rotate by β then by α = Rotate by $\alpha+\beta$.

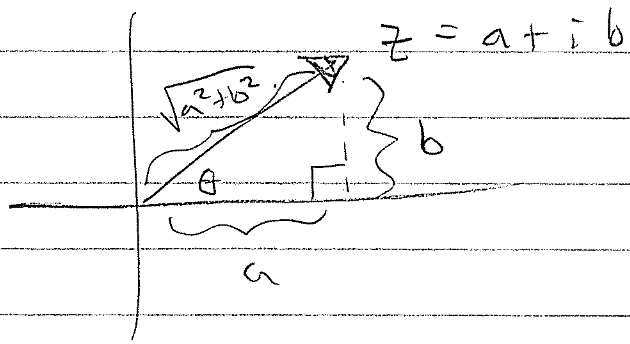
Corollary:

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

"$\operatorname{cis}(\alpha)\operatorname{cis}(\beta) = \operatorname{cis}(\alpha+\beta)$".

$\operatorname{cis} : \mathbb{R} \to \mathbb{C}$
sends addition $\to$ multiplication.
VERY USEFUL.

The polar form of $z \in \mathbb{C} = \mathbb{R}^2$.



$$a + ib = \sqrt{a^2+b^2}\left(\left(\frac{a}{\sqrt{a^2+b^2}}\right) + i\left(\frac{b}{\sqrt{a^2+b^2}}\right)\right)$$

$$a + ib = |z|(\cos\theta + i\sin\theta)$$
$$|z|\operatorname{cis}(\theta)$$

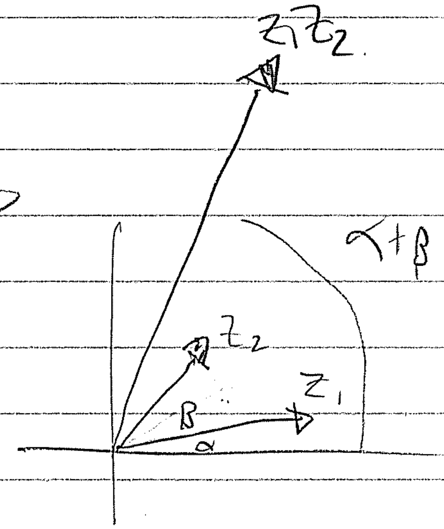
Cartesian form

polar form.

To multiply $z_1, z_2 \in \mathbb{C}$.

$$z_1 z_2 = |z_1|\,\mathrm{cis}(\theta_1)\,|z_2|\,\mathrm{cis}(\theta_2)$$
$$= |z_1||z_2|\,\mathrm{cis}(\theta_1 + \theta_2)$$

- lengths <u>multiply</u>
- angles <u>add</u>



Now we can SOLVE EQUATIONS.

Eg. Solve $x^n - 1 = 0$ (i.e. $x^n = 1$)
"find all <u>nth roots</u> of <u>unity</u>"

Note $1 = \mathrm{cis}(2\pi k)$
$= \cos(2\pi k) + i\sin(2\pi k)$ for any $k \in \mathbb{Z}$
$\quad 1 \quad + i \quad 0$.

Hence we have $\left(\mathrm{cis}\left(\frac{2\pi k}{n}\right)\right)^n = \mathrm{cis}\left(n\frac{2\pi k}{k}\right)$
$= 1$

$\mathrm{cis}\left(\frac{2\pi k}{n}\right) = x$ is a solution $\forall k \in \mathbb{Z}$.
HOW MANY SOLUTIONS?

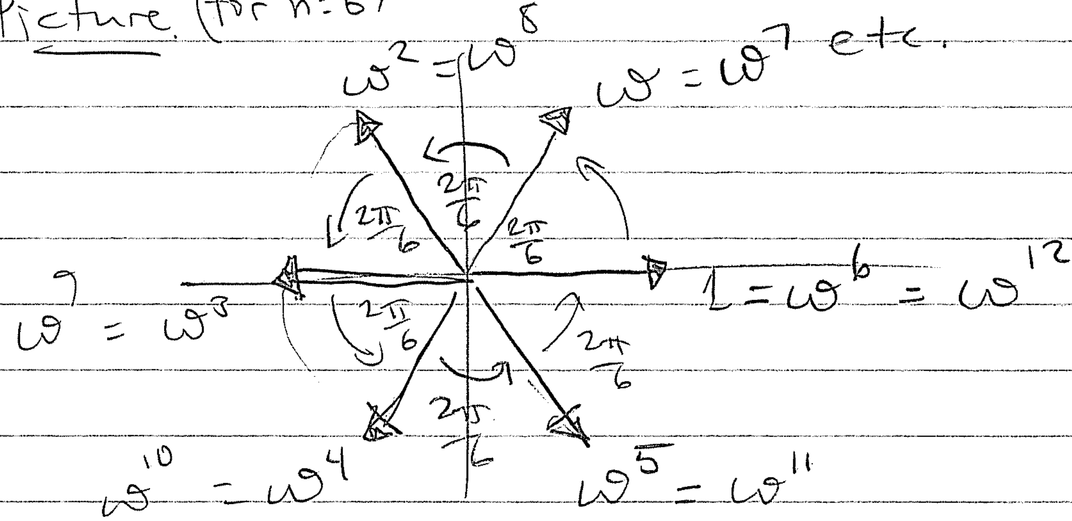
Let $\omega = \text{cis}\left(\frac{2\pi}{n}\right)$. so

$\omega^k = \text{cis}\left(\frac{2\pi k}{n}\right)$.

Get a bunch of solutions

$x = 1, \omega, \omega^2, \omega^3, \omega^4, \ldots$

Picture (for $n=6$).



for general n,

$1, \omega, \omega^2, \ldots, \omega^{n-1}$

~~form~~ vertices of regular n-gon

are the n

So $x^n - 1 = 0$ has at least n solutions.

Could it have more? **NO**

Summary: Equation $x^n - 1 = 0$ has exactly n solutions

$$\underbrace{1, \omega, \omega^2, \ldots, \omega^{n-1}}_{\text{vertices of regular } n\text{-gon}}$$

"the n^{th} roots of unity"

where $\omega = \text{cis}(2\pi/n)$

and hence

$$\omega^k = \left(\text{cis}\left(\frac{2\pi}{n}\right)\right)^k = \text{cis}\left(\frac{2\pi k}{n}\right)$$

$$\boxed{\omega^k = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)}$$

Factorization:

$$\boxed{x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1})}$$

Cool Fact:

$$(x^n - 1) = (x-1)(1 + x + x^2 + \cdots + x^{n-1}) \checkmark$$
$$= (x-1)(x-\omega)\cdots(x-\omega^{n-1})$$

Hence

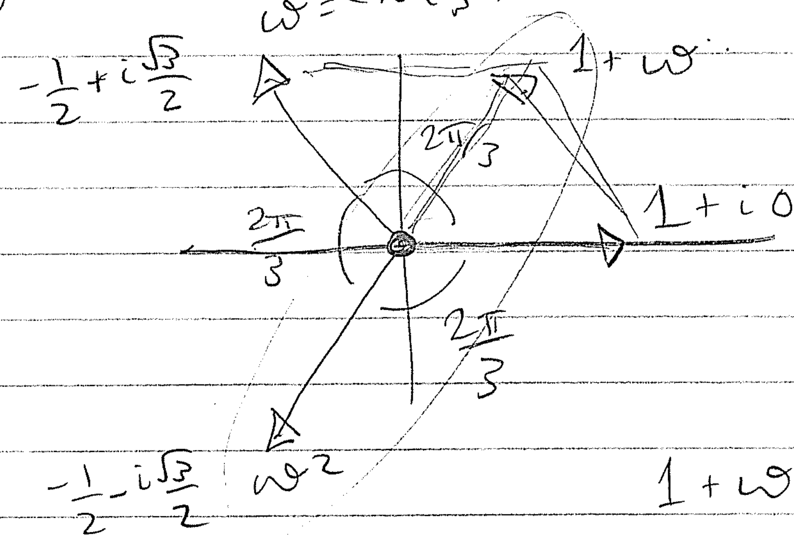
$$1 + x + x^2 + \cdots + x^{n-1} = (x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1})$$

Put $x = \omega$ to get

$$\boxed{1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0}$$

"center of mass"

eg $n=3$. $\omega = \text{cis}\left(\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$



$1+\omega = -\omega^2$

$$1+\omega+\omega^2$$

$$= (1+i0) + \left(-\frac{1}{2} + \cancel{\frac{i\sqrt{3}}{2}}\right) + \left(-\frac{1}{2} - \cancel{\frac{i\sqrt{3}}{2}}\right)$$

$$= 1 - \frac{1}{2} - \frac{1}{2} = 0 \quad ✓$$

Another eg. SQUARE ROOT OF i

Solve $x^2 - i = 0$. (i.e $x^2 = i$)

Find the square roots of i

Suppose $x = r\,\text{cis}\,\theta$

Then $x^2 = r^2\,\text{cis}(2\theta) = i = 1\,\text{cis}\left(\frac{\pi}{2}\right)$

$r^2 \operatorname{cis}(2\theta) = 1 \operatorname{cis}\left(\frac{\pi}{2}\right)$.

But $r \geq 0$

Get two equations: $r^2 = 1 \Rightarrow r = \pm 1$.

$\operatorname{cis}(2\theta) = \operatorname{cis}\left(\frac{\pi}{2}\right)$.

$$\operatorname{cis}(2\theta) = \operatorname{cis}\left(\frac{\pi}{2}\right).$$

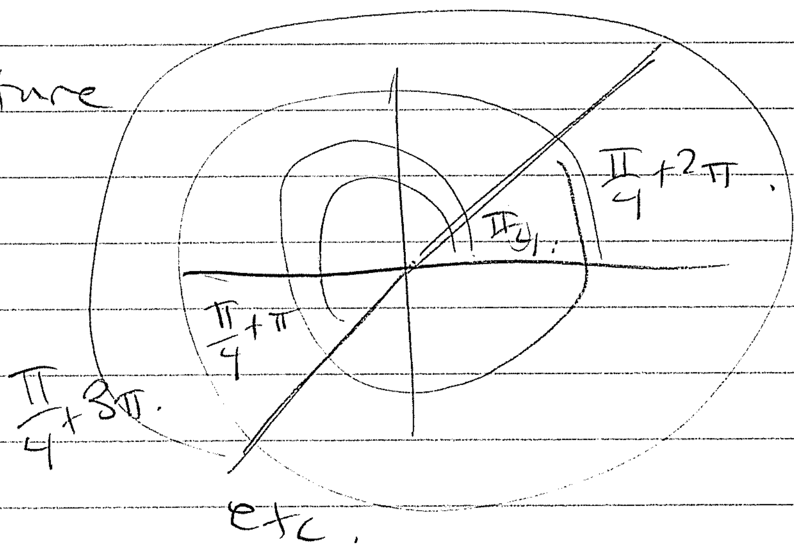
$$\Downarrow$$

$$2\theta = \frac{\pi}{2} + 2\pi k \quad \text{for any } k \in \mathbb{Z}$$

Same angle.

$$\theta = \frac{\pi}{4} + \pi k, \quad \forall k \in \mathbb{Z}.$$

Picture



Solution

$$\sqrt{i} = 1 \operatorname{cis}\left(\frac{\pi}{4}\right) \text{ OR } 1 \operatorname{cis}\left(\frac{\pi}{4} + \pi\right)$$

$$\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) \text{ OR } \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)$$

HW 2 due Fri.

Office Hours.

Today & Tomorrow 2:30 – 4:00

Exam 1; Fri Feb 18.

Recall: ROOTS OF UNITY. $\sqrt[n]{1}$

Let $\omega = \operatorname{cis}\left(\frac{2\pi}{n}\right)$, so $\omega^k = \operatorname{cis}\left(\frac{2\pi k}{n}\right)$

"de Moivre".

Then for __any__ $k \in \mathbb{Z}$ we have

$(\omega^k)^n = \left[\operatorname{cis}\left(\frac{2\pi}{n}k\right)\right]^n = \operatorname{cis}\left(\frac{2\pi k \cancel{n}}{\cancel{n}}\right) = \operatorname{cis}(2\pi k)$

de Moivre.

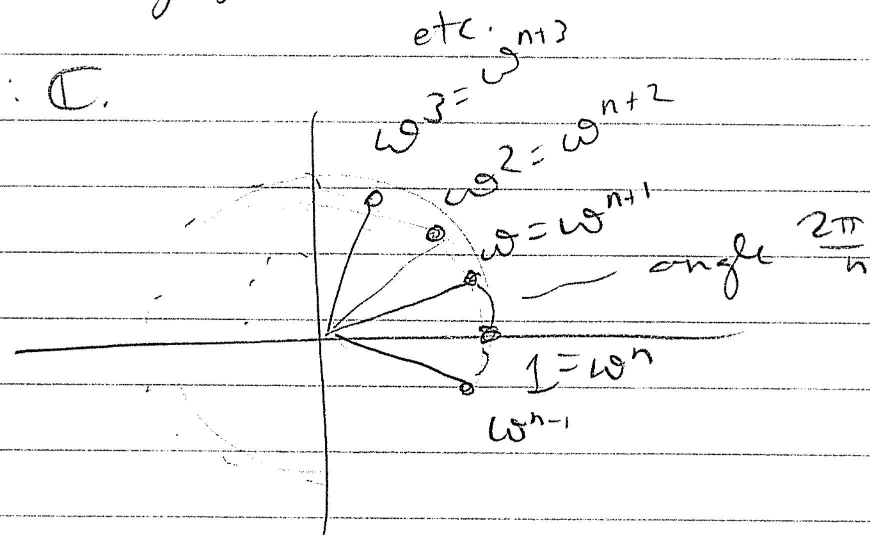
$= \underline{1}$

So $1, \omega, \omega^2, \omega^3, \ldots$

are all n^{th} roots of 1.

How many?

__Picture__: $\mathbb{C}$.



Get *exactly* n solutions to $x^n - 1 = 0$,

$$\sqrt[n]{1} = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$$

"the nth roots of unity"

vertices of a regular n-gon in $\mathbb{C}$.

Corollary:

$$x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1})$$

Proof: FACTOR THEOREM $\boxtimes$

Corollary:

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$$

"n-gon is centered at 0"

Proof: We know

$$(x^n - 1) = (x-1)(1 + x + x^2 + \cdots + x^{n-1})$$

for *any* x. Put $x = \omega$. $\boxtimes$

Problem: Compute $\sqrt{i}$. How?

Let $x^2 = i$

Polar form $x = r\,\text{cis}\,\theta$, $r \geq 0$.

Solve for r, θ.

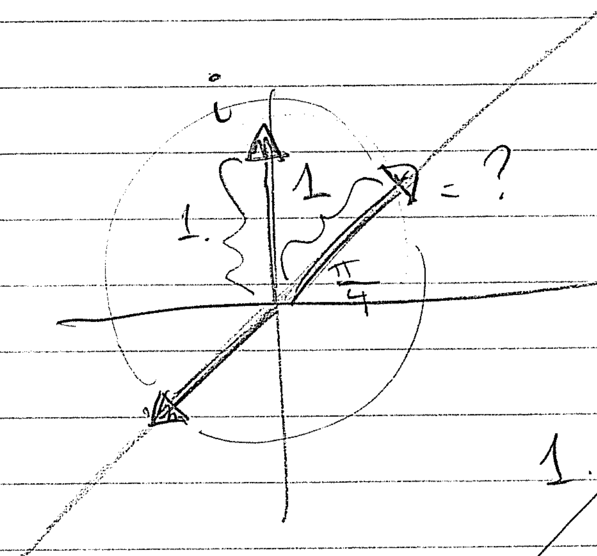
$$x^2 = r^2 \text{cis}(2\theta) = i = 1\,\text{cis}\left(\frac{\pi}{2}\right)$$

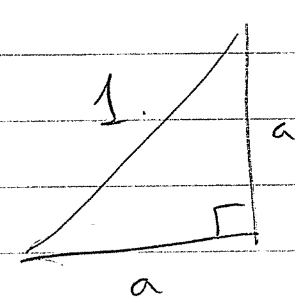
Hence $r^2 = 1 \implies r = 1$.

$$2\theta = \frac{\pi}{2} + 2\pi k \quad \text{for any } k \in \mathbb{Z}.$$

$$\boxed{\theta = \frac{\pi}{4} + \pi k} \quad \text{for any } k \in \mathbb{Z}.$$

How many choices? 2





$a^2 + a^2 = 1^2$

$\Rightarrow a = \sqrt{\frac{1}{2}}$.

Summary:

$$\sqrt{i} = \left\{ \sqrt{\tfrac{1}{2}} + i\sqrt{\tfrac{1}{2}} \;,\; -\sqrt{\tfrac{1}{2}} - i\sqrt{\tfrac{1}{2}} \right\}$$

two square roots.

In General: Given $z \in \mathbb{C}$, find $\sqrt[n]{z}$.

Let $z = r\,\mathrm{cis}\,\theta$, $r \geq 0$.
Let $\omega = \mathrm{cis}\left(\frac{2\pi}{n}\right)$.
Let $\sqrt[n]{r}$ = <u>positive</u>, <u>real</u> nth root.

Then

<u>Theorem.</u>

$$\sqrt[n]{z} = \left\{ \sqrt[n]{r}\,\mathrm{cis}\left(\frac{\theta}{n}\right)\omega^{i} \;:\; i = 0, 1, 2, \ldots, n-1 \right\}$$

<u>Proof:</u>

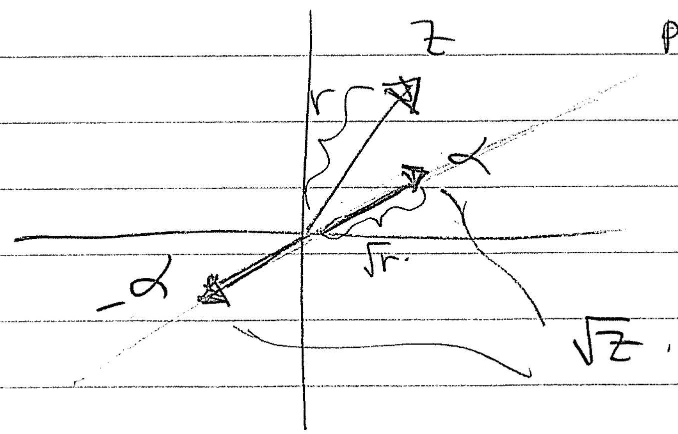
$$\left(\sqrt[n]{r}\,\mathrm{cis}\left(\frac{\theta}{n}\right)\omega^{i}\right)^{n} = r\,\mathrm{cis}\,\theta\,\Big(\overset{1}{\cancel{\omega^{n}}}\Big)^{i} = r\,\mathrm{cis}\,\theta = z.$$

we get n solutions to $x^2 - z = 0$.
There can be no more (Factor Theorem) ∎

<u>Picture</u> : $\sqrt[n]{z}$ = regular n-gon with one point at $\sqrt[n]{r}\,\text{cis}\left(\frac{\theta}{n}\right)$.

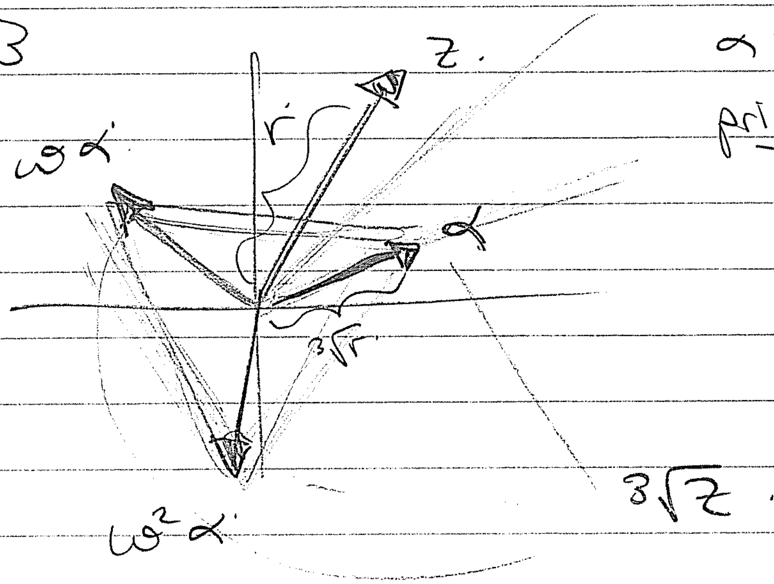
<u>eg</u>. $n=2$.

$\alpha = \sqrt{r}\,\text{cis}\left(\frac{\theta}{2}\right)$
<u>principal</u> square root.

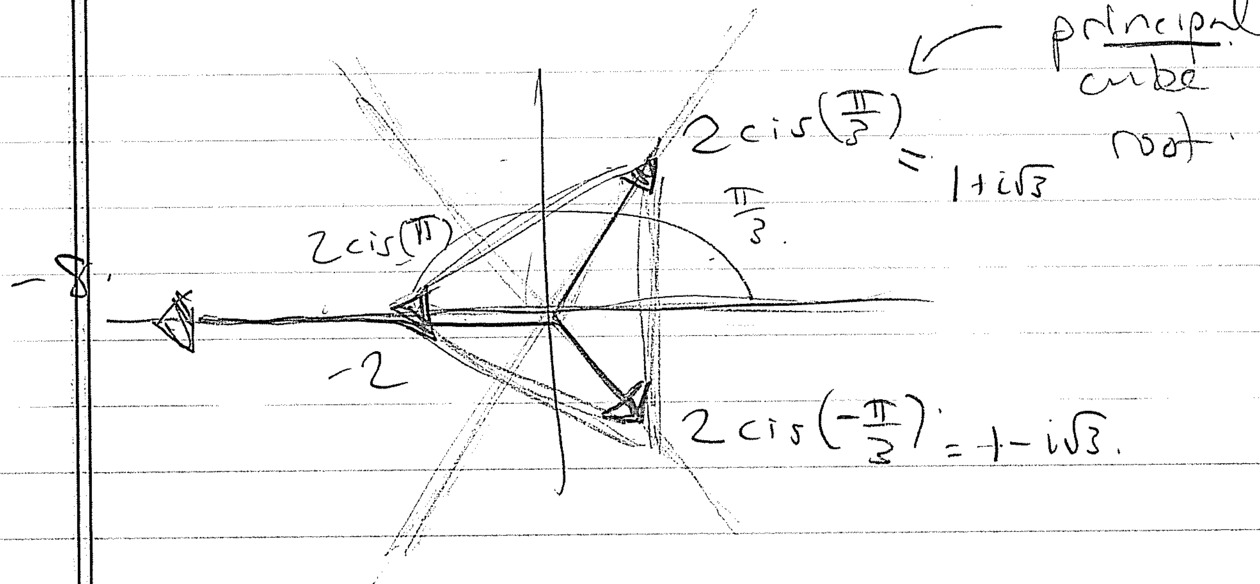


$n=3$

$\alpha = \sqrt[3]{r}\,\text{cis}\left(\frac{\theta}{3}\right)$
<u>principal</u> <u>cube</u> root.



<u>Eg</u> Cube roots of -8.



$$\mathrm{cis}\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Hence

$$\sqrt[3]{-8} = \left\{-2,\ 1+i\sqrt{3},\ 1-i\sqrt{3}\right\}$$

<u>Exercise</u>: Try $\sqrt[4]{-4}$.

To solve: $ax^2 + bx + c = 0$ when $a, b, c \in \mathbb{C}$

$$x = \frac{-b + \alpha}{2a} \quad \text{where} \quad \alpha \in \sqrt{b^2 - 4ac}$$

HW 2 due NOW

Exam 1, next Friday.

Today:

Complex Conjugation

$a+ib \mapsto a-ib$.

Recall: If we think of x^n-1 as an element of $\mathbb{C}[x]$ we can factor:

$$x^n-1 = (x-1)(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1})$$

it factors completely (it "splits").

where $\omega = \text{cis}\left(\frac{2\pi}{n}\right)$.

We say x^n-1 "splits" over $\mathbb{C}$.

But what if we think of $x^n-1 \in \mathbb{R}[x]$

How much does it factor over $\mathbb{R}$?

Idea: Define the conjugate map.

for $z \in \mathbb{C}$, $z = a+ib$, let

$$\overline{z} := a - ib$$

Useful properties: ~~for $z, w \in \mathbb{C}$ we have~~

- $\overline{zw} = \overline{z}\,\overline{w}$ (Exercise).
- $\overline{z+w} = \overline{z} + \overline{w}$

We say conjugation is a field automorphism

$$\mathbb{C} \to \mathbb{C}$$
$$z \mapsto \bar{z}$$

preserves field structure.

Theorem: Let $f(x) \in \mathbb{R}[x]$.

If f has a complex root $f(z) = 0$ for some $z \in \mathbb{C}$ then $\bar{z}$ is also a root, i.e. $f(\bar{z}) = 0$.

Proof: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ for some $a_0, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$.
We assume that $f(z) = 0$, or

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

Take conjugate on both sides.

$$\overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \overline{0}$$

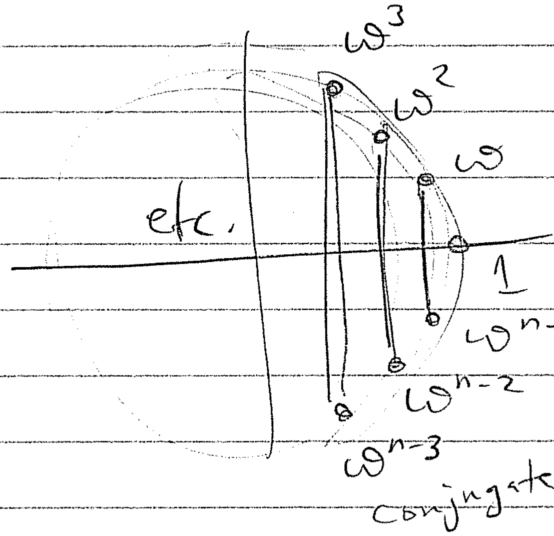
$$a_n (\bar{z})^n + a_{n-1} (\bar{z})^{n-1} + \cdots + a_1 \bar{z} + a_0 = 0$$

Hence $f(\bar{z}) = 0$ as desired. ▨

In words: complex roots of $f(x) \in \mathbb{R}[x]$ come in conjugate pairs

Corollary: The number of complex roots is $\underline{\underline{\text{even}}}$.

Example: The $\mathbb{C}$-roots of $x^n - 1 \in \mathbb{R}[x]$ are



Note: $\omega^{n-k} = \omega^{-k} := \dfrac{1}{\omega^k}$

Because $\omega^{n-k}\omega^k = \omega^{n-\cancel{k}+\cancel{k}} = 1$.

And: $\omega^{-k} = \overline{\omega^k}$.

Because $\omega^{-k} = \dfrac{1}{\omega^k} = \dfrac{1}{\omega^k}\dfrac{\overline{\omega^k}}{\overline{\omega^k}} = \dfrac{\omega^k}{|\omega^k|^2_{1}} = \omega^k$.

Useful Properties: for $z = a + ib$

- $z\overline{z} = |z|^2 = a^2 + b^2 \in \mathbb{R}$ (Exercise)
- $z + \overline{z} = 2a \in \mathbb{R}$

Together:

$$(x-z)(x-\overline{z}) = x^2 - (z+\overline{z})x + z\overline{z}$$
$$= x^2 - 2ax + |z|^2$$

Example: $\omega^k = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$

So $\omega^k \overline{\omega^k} = \omega^k \omega^{-k} = \omega^0 = 1$.

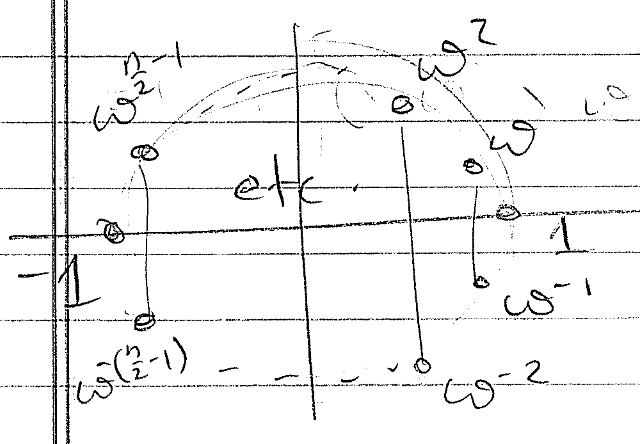
$$\omega^k + \omega^{-k} = 2\cos\left(\frac{2\pi k}{n}\right)$$

$$\boxed{(x-\underbrace{\omega^k}_{\text{Complex}})(x-\omega^{-k}) = \underbrace{x^2 - 2\cos\left(\frac{2\pi k}{n}\right)x + 1}_{\text{Real!}}}$$

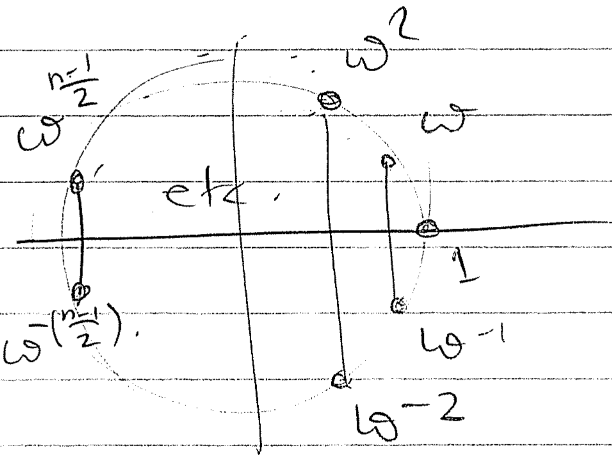
Finally: We can FACTOR $x^n - 1$ over $\mathbb{R}$.

2 cases

n even



n odd



n even:

$$x^n - 1 = (x-1)(x+1)\left[(x-\omega)(x-\omega^{-1})\right]\left[(x-\omega^2)(x-\omega^{-2})\right] \cdots \left[(x-\omega^{\frac{n}{2}-1})(x-\omega^{-\frac{n}{2}+1})\right]$$

$$= (x-1)(x+1)\prod_{k=1}^{\frac{n}{2}-1}\left[(x-\omega^k)(x-\omega^{-k})\right]$$

$x^n -$

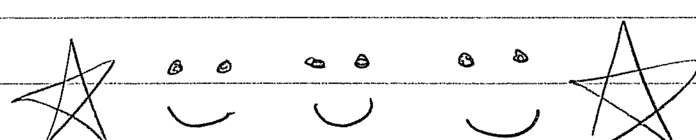
$$= (x-1)(x+1)\underbrace{\prod_{k=1}^{\frac{n}{2}-1}\left(x^2 - 2\cos\left(\frac{2\pi k}{n}\right)x + 1\right)}_{\text{factorization into "primes" over } \mathbb{R}}$$

n odd:

$$x^n - 1 = (x-1)\prod_{k=1}^{\frac{n-1}{2}}\left(x^2 - 2\cos\left(\frac{2\pi k}{n}\right)x + 1\right)$$

cannot factor further

$$x^n - 1 = \begin{cases} (x-1)(x+1)\displaystyle\prod_{k=1}^{\frac{n}{2}-1}\left(x^2 - 2\cos\left(\frac{2\pi k}{n}\right)x + 1\right) & n \text{ even} \\ (x-1)\displaystyle\prod_{k=1}^{\frac{n-1}{2}}\left(x^2 - 2\cos\left(\frac{2\pi k}{n}\right)x + 1\right) & n \text{ odd} \end{cases}$$



3 smiling faces & 2 stars