MTH 461 F Spring 2011

Syllabus: 6 HW, 3 in-class exams

HW	25%	
E1	25%	Fri Feb 18
E2	25%	Fri Mar 25
E3	25%	Fri Apr 29

Text

I have 3 goals for MTH461

- follow-up to 230
- prelude to 561–562
- terminal course in Algebra

can we do it?

What is Algebra?

prehistory $\longrightarrow$ MTH 461, mid 1800's $\longrightarrow$ MTH 561–562

study of polynomial equations.

study of abstract structures.

e.g. groups
rings
fields
modules
categories
etc . . . .

"Concrete".

MTH 461 = Pre-Abstract Algebra through <u>history</u>

Let's begin.

Robert Recorde, 1557.

Book: "The Whetstone of Witte".

$+$, $-$, $=$

"noe 2 thynges can be moare equalle"

Francois Viète, 1591

"Intro. to Analytic Art".

used letters to denote <u>unknown</u> quantities.

Consonants B, C, D, F, .. for parameters

vowels A, E, I, O, U, .. for variables

Renée Descartes, 1637

"La Géométrie".

a, b, c, .. for parameters

x, y, z, .. for variables.

↓

<u>modern</u> notation.

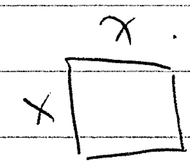
"Algebra" was born in 830

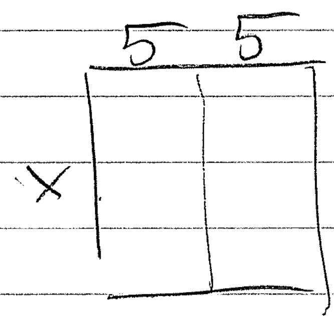
al-Khwarizmi "al-jabr w al-muqabalah"
"science of restoring & opposition"

"al-jabr" = "Algebra"
"al-Khwarizmi" = "algorithm"

Eg. Solve $x^2 + 10x = 39$ for x.

Solution.

$x^2 =$ 

$10x =$ 

Put together

$$\begin{array}{c|c|c|} & x & 5 \\ \hline x & x^2 & 5x \\ \hline 5 & 5x & 25 \\ \hline \end{array}$$

Area of big square

$$= \underbrace{x^2 + 5x + 5x} + 25$$

$$= 39 + 25 = 64 = 8^2$$

Hence its side length is

$$x + 5 = 8$$
$$x = 2$$

DONE.

In modern notation:

Solve $ax^2 + bx + c = 0$ for x.

Assuming $a \neq 0$, divide by a to get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Picture

	x	$b/2a$
x	x^2	$\frac{b}{2a}x$
$b/2a$	$\frac{b}{2a}x$	$\frac{b^2}{4a^2}$

"complete the square"

$$\text{Area} = \underbrace{x^2 + \frac{b}{a}x} + \frac{b^2}{4a^2}$$

$$= -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{-4ac}{a} + \frac{b^2}{4a^2}$$

$$= \boxed{\frac{b^2-4ac}{4a^2}}$$

Hence. Side Length is

$$x + \frac{b}{2a} = \sqrt{\frac{b^2-4ac}{4a^2}}.$$

$$x = -\frac{b}{2a} + \frac{1}{2a}\sqrt{b^2-4ac}$$

$$= \frac{-b + \sqrt{b^2-4ac}}{2a}.$$

al-Khwarizmi:
- used <u>words</u>, not symbols
- did <u>not</u> accept negative "numbers"

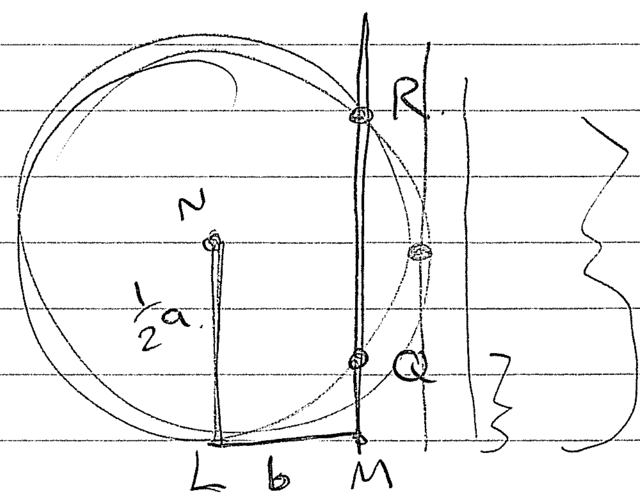
$$\left.\begin{aligned} x^2 + px &= q \\ x^2 + q &= px \\ x^2 &= px + q \end{aligned}\right\}$$ 3 separate solutions.

Another Problem:

What happens when $b^2 - 4ac < 0$?

Descartes' Answer:

Intersect a line & circle



HW

Then $\overline{MQ}$ $\overline{MR}$ are solutions to

$$z^2 = az - b^2 \qquad (a, b > 0)$$

$$z^2 - az + b^2 = 0.$$

i.e.

$$z = \frac{a \pm \sqrt{a^2 + 4b^2}}{2}.$$

for $0 < b < \frac{1}{2}a$ — 2 solutions

$b = \frac{1}{2}a$ — 1 solution

$\frac{1}{2}a < b$ — NO SOLUTION.

i.e. for $a^2-4b^2<0$,

$\sqrt{a^2-4b^2}$ is "impossible".

HW 1 due next Fri Jan 29.

The general quadratic equation:

Consider $a, b, c \in \mathbb{R}$ = "real numbers".
Solve for x in

$$ax^2 + bx + c = 0, \quad a \neq 0.$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

TRICK

$$x^2 + \frac{b}{a}x + \left(\frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + \frac{c}{a} = 0$$

$$\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right) = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Issue: What does $\sqrt{a}$ mean?

$\alpha > 0$

If $\alpha \in \mathbb{R}$, ~~$\alpha < 0$~~ then $\sqrt{\alpha}$ is <u>not</u> a real number.

It is a <u>pair</u> of numbers

$$\sqrt{\alpha} = \{+\alpha, -\alpha\}$$

STRANGE.

By convention we usually say $\sqrt{\alpha} := +\alpha$.

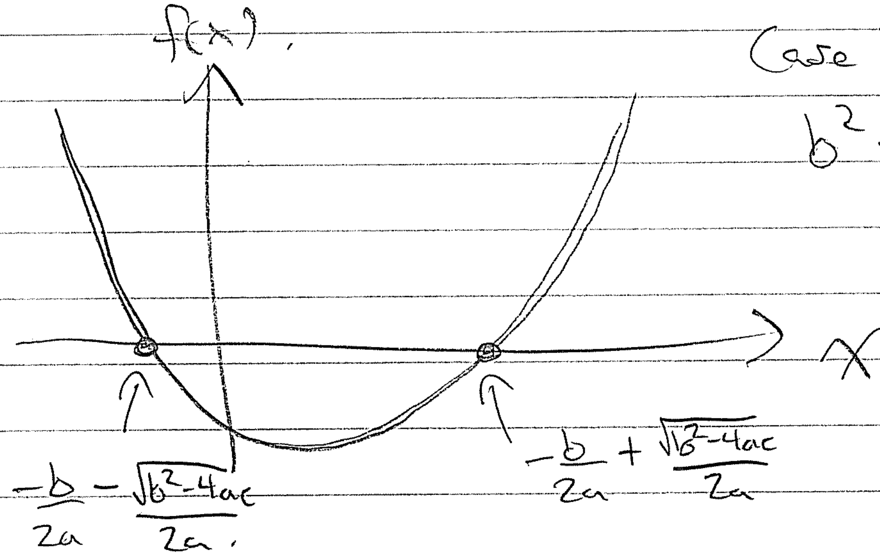
Then the quadratic has 2 solutions (roots)

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ OR } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

<u>Visualize</u> : let $f(x) = ax^2 + bx + c$.

Case I:

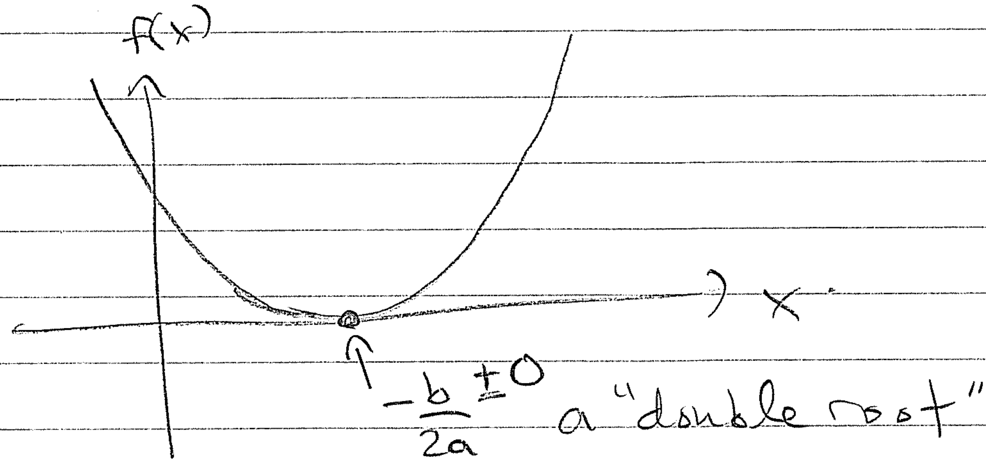
$b^2 - 4ac > 0$.



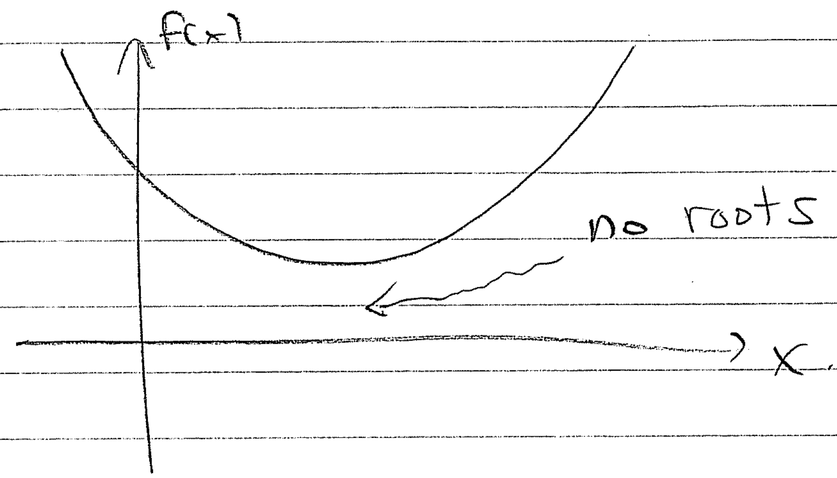
Case II:

If $\boxed{b^2 - 4ac} = 0$ then

the "discriminant".



Case III: $b^2 - 4ac < 0$



Everyone said

$\sqrt{\text{negative}}$ = doesn't exist
= "impossible"
= "an amphibian between being and nonbeing" — Leibniz

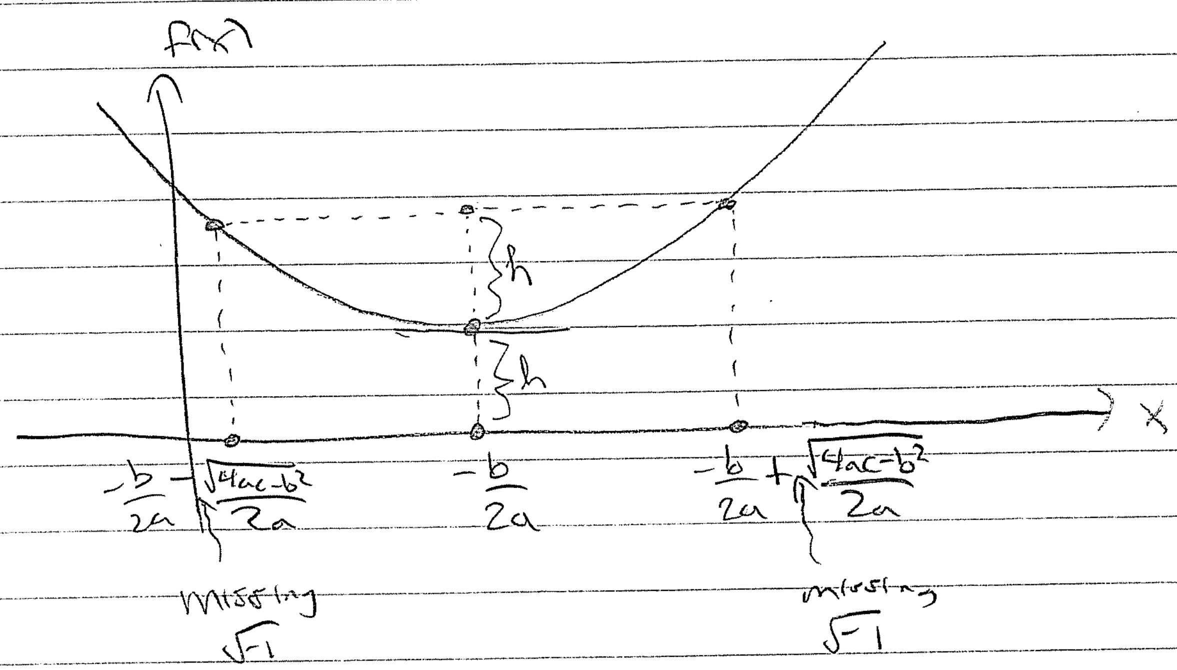
Contrary to history, let's get fancy.

If $b^2-4ac < 0$ then

$$\sqrt{b^2-4ac} = \sqrt{(-1)(4ac-b^2)}$$
$$= \underset{?}{\sqrt{-1}}\ \underset{\in \mathbb{R}}{\sqrt{4ac-b^2}}$$

Roots: $x = \dfrac{-b}{2a} \pm \sqrt{-1}\ \dfrac{\sqrt{4ac-b^2}}{2a}$.

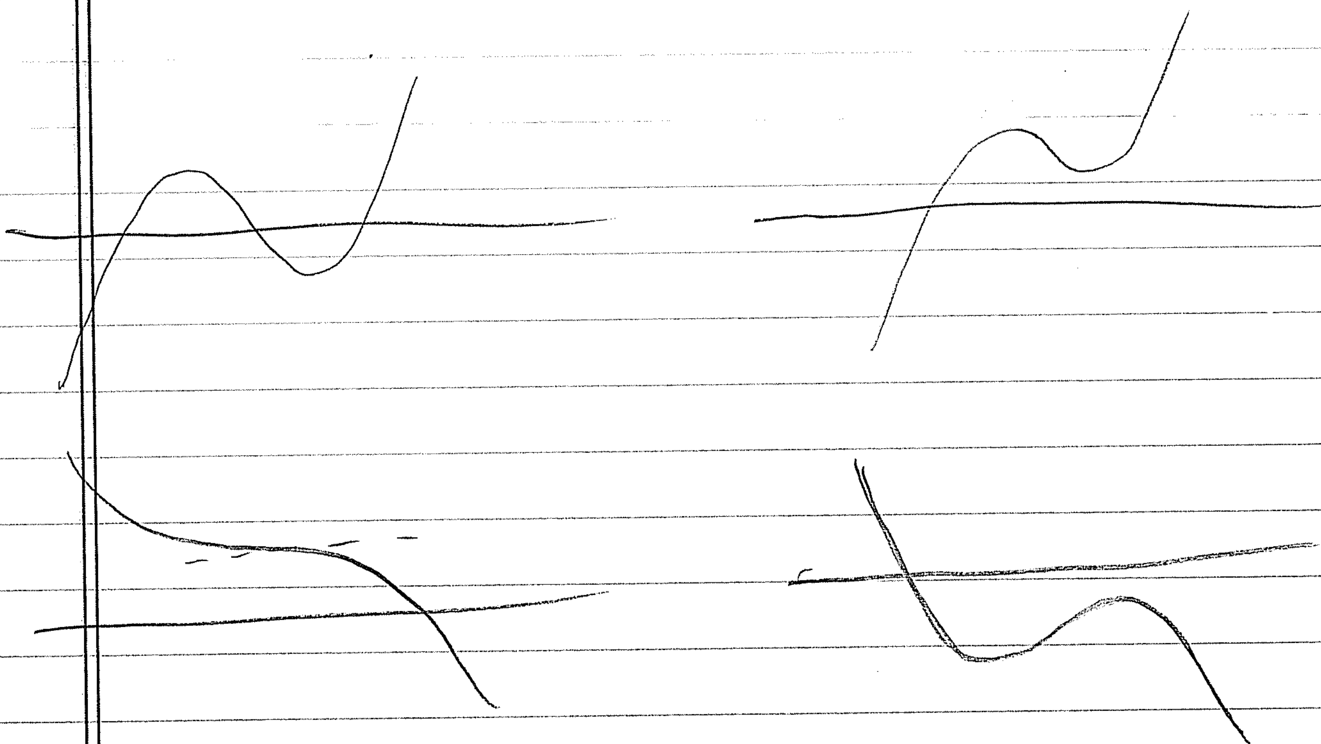
Picture:



(Check!)

Next Step: The <u>cubic</u> equation

$$f(x) = ax^3 + bx^2 + cx + d = 0.$$



<u>Theorem</u>: $f(x) = 0$ has ≥ 1 real root.

<u>Proof</u>: Suppose $a > 0$. Then $\lim_{x \to \infty} f(x) = +\infty$ and $\lim_{x \to -\infty} f(x) = -\infty$. By Intermediate Value Theorem, f must <u>cross</u> x-axis. $\blacksquare$

Call the real root $\alpha \in \mathbb{R}$.

Theorem (Descartes, 1637 Again!)

If $f(\alpha)=0$ then $f(x) = (x-\alpha)\, g(x)$

quadratic.

Proof: for any $n \in \mathbb{N} = \{0,1,2,3,\dots\}$.
we have

$$x^n - \alpha^n = (x-\alpha)\left(x^{n-1} + x^{n-2}\alpha + x^{n-3}\alpha^2 + \dots + \alpha^{n-1}\right)$$

"difference of nth powers".

Then $f(x) = f(x) - \cancelto{0}{f(\alpha)}$.

$$= (ax^3+bx^2+cx+d) - (a\alpha^3+b\alpha^2+c\alpha+d)$$

$$= a(x^3-\alpha^3) + b(x^2-\alpha^2) + c(x-\alpha) + \cancelto{0}{(d-d)}$$

$$= a(x-\alpha)(x^2+x\alpha+\alpha^2) + b(x-\alpha)(x+\alpha) + c(x-\alpha)$$

$$= (x-\alpha)\Big[\underbrace{a(x^2+x\alpha+\alpha^2) + b(x+\alpha) + c}_{\text{quadratic}}\Big]$$

$\blacksquare$

HW 1 due Friday Jan 28

Office Hours
2:30 – 4:00 P
Wed & Thurs.

Let $\mathbb{R}[x]$ denote the set of polynomials

$$f(x) = a_n x^{n} + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$
$a_n \neq 0$.
Say $\text{degree}(f) = n$.

Big Theorem:

The Factor Theorem (Descartes, 1637).

Consider a polynomial $f(x) \in \mathbb{R}[x]$ of degree n. If $\alpha \in \mathbb{R}$ is a root of f (i.e. $f(\alpha) = 0$) then we can write

$$f(x) = (x - \alpha)\, g(x)$$

where $g(x) \in \mathbb{R}[x]$ has degree $n-1$.

<u>Proof</u>: For any $b = 1, 2, 3, \ldots$ we have

$$(x^b - \alpha^b) = (x - \alpha)\underbrace{\left(x^{b-1} + x^{b-2}\alpha + x^{b-2}\alpha^2 + \cdots + x\alpha^{b-2} + \alpha^{b-1}\right)}_{\text{degree } b-1.}$$

call this $\varphi_{b-1}(x, \alpha)$

Then we have

$$f(x) = f(x) - \overset{0}{\cancel{f(\alpha)}}$$

$$= (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) - (a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1 \alpha + a_0)$$

$$= a_n (x^n - \alpha^n) + a_{n-1}(x^{n-1} - \alpha^{n-1}) + \cdots + a_1(x - \alpha) + \overset{0}{\cancel{a_0 - a_0}}$$

$$= (x - \alpha)\underbrace{\left[a_n \varphi_{n-1}(x,\alpha) + a_{n-1}\varphi_{n-2}(x,\alpha) + \cdots + a_2\varphi_1(x,\alpha) + a_1\right]}_{\text{poly of deg } n-1}$$

$\blacksquare$

<u>Corollary</u>: Let $f(x) \in \mathbb{R}[x]$ have deg n. Then $f(x) = 0$ has <u>at most n</u> real solutions.

Proof: If $f(x)=0$ has zero roots, done ✓

So suppose $f(\alpha)=0$ for some $\alpha \in \mathbb{R}$.

Then

$$f(x) = (x-\alpha)\underbrace{g(x)}_{\text{deg } n-1}$$

Now suppose $\beta \in \mathbb{R}$ is another solution.

i.e. suppose $f(\beta) = \underbrace{(\beta-\alpha)}_{\neq 0} g(\beta) = 0$

Then $g(\beta)=0$. But by induction, $g(x)=0$ has $\leq n-1$ solutions.

So $f(x)=0$ has $\leq 1+(n-1)$ solutions $\blacksquare$

ENOUGH RIGOR FOR TODAY?

eg. let $f(x) = x^3-7x^2+8x-2$

and NOTE that $f(1)=0$. Hence we can divide

$$
\begin{array}{r|l}
 & x^2 - 6x + 2 \quad \longrightarrow \text{quotient} \\
\hline
x-1 & x^3 - 7x^2 + 8x - 2 \\
 & x^3 - x^2 \\
\hline
 & 0 - 6x^2 + 8x - 2 \\
 & \quad -6x^2 + 6x \\
\hline
 & \qquad 2x - 2 \\
 & \qquad 2x - 2 \\
\hline
 & \qquad\quad 0 \quad \text{— remainder.}
\end{array}
$$

We have

$$f(x) = (x-1)(x^2-6x+2)$$

So $f(x) = 0$ when

$x = 1$ OR $x^2 - 6x + 2 = 0$

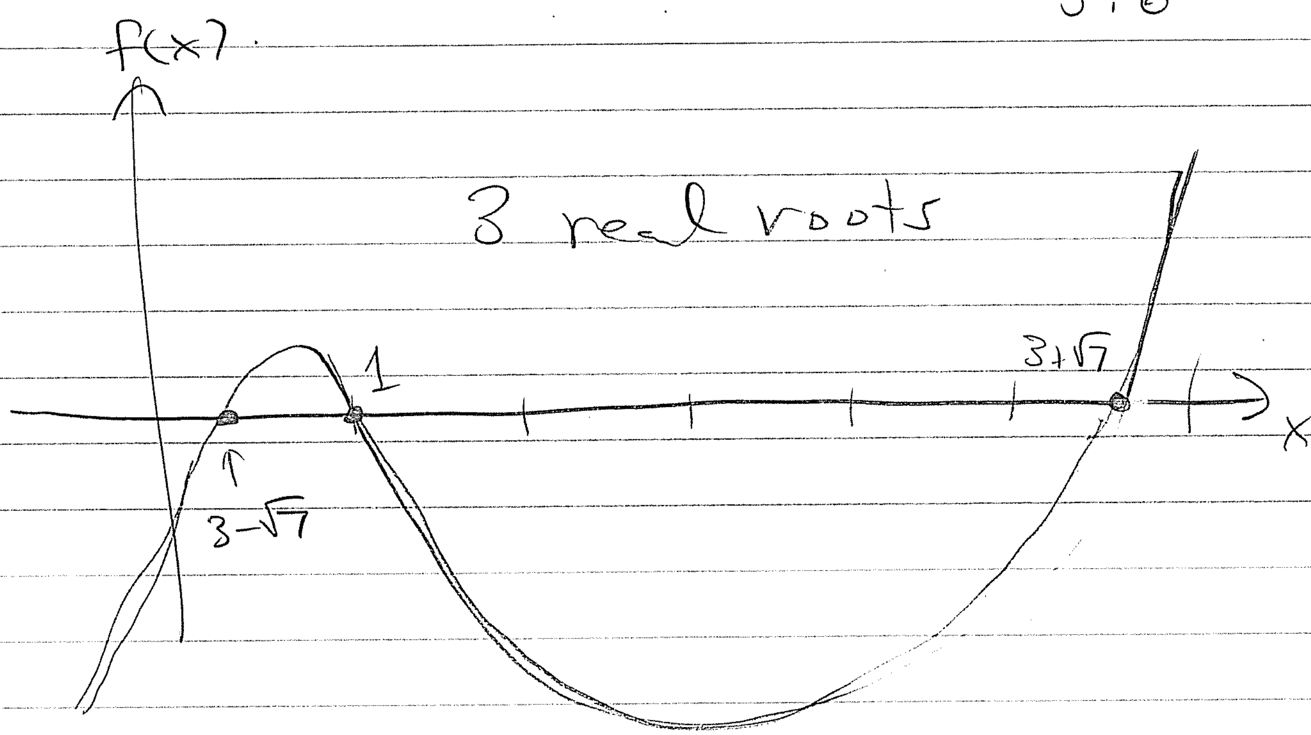
$$x = \frac{6 \pm \sqrt{36-8}}{2}$$

$$= \frac{6 \pm \sqrt{4 \cdot 7}}{2} = \frac{6 \pm \sqrt{4}\sqrt{7}}{2}$$

$$= 3 \pm \sqrt{7}$$

≈ 0.4 OR 5.6

Pic.



Moral: Given a cubic $f(x)$, if we can find 1 root, then we have them all!

Q: Can we find 1 root?

Story:
Scipione del Ferro (Bologna, died 1526)
Antonio Maria Fior, student
Niccolò Tartaglia, Feb 12, 1535
Girolamo Cardano, Tartaglia visited in 1539.

Cardano "Ars Magna" 1545.
the great art

Cardano's Formula

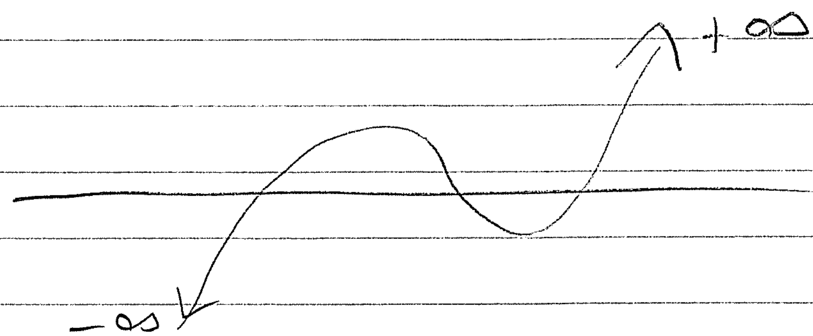
The equation $x^3 + px + q = 0$ has solution

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Something NEW!

HW 1 due Friday
Office Hours
Wed & Thurs 2:30 — 4:00 P

Recall:
- every cubic has a real root



- if we can find a root, then we have them all

if $f(x)$ has root r, divide by $(x-r)$ to get $f(x) = (x-r)\,g(x)$. Then use quadratic formula to solve $g(x)$.

- <u>Can</u> we find always a root? (exactly).

del Ferro → Fior X Tartaglia
↓
Cardano

Brace yourselves!

Here is Cardano's solution.

To solve $ax^3 + bx^2 + cx + d = 0$
with $a, b, c, d \in \mathbb{R}$, $a \neq 0$.

First divide by a,

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$$

<u>TRICK 1</u>: Substitute $x = y + \alpha$ for some $\alpha \in \mathbb{R}$.

$$(y+\alpha)^3 + \frac{b}{a}(y+\alpha)^2 + \frac{c}{a}(y+\alpha) + \frac{d}{a} = 0.$$

$$y^3 + 3y^2\alpha + 3y\alpha^2 + \alpha^3 + \frac{b}{a}\left(y^2 + 2y\alpha + \alpha^2\right) + \frac{c}{a}(y+\alpha) + \frac{d}{a} = 0$$

$$y^3 + \cancel{\left(3\alpha + \frac{b}{a}\right)}y^2 + \left(3\alpha^2 + \frac{2b}{a}\alpha + \frac{c}{a}\right)y$$
$$+ \left(\alpha^3 + \frac{b}{a}\alpha^2 + \frac{c}{a}\alpha + \frac{d}{a}\right) = 0.$$

Which α is good? Take $\alpha = -\frac{b}{3a}$ to get

$$y^3 + 0y^2 + py + q = 0.$$

~~The "depressed" cubic~~.

with
$$\left.\begin{aligned} p &= 3\alpha^2 + \frac{2b\alpha}{a} + \frac{c}{a} \\ q &= \alpha^3 + \frac{b}{a}\alpha^2 + \frac{c}{a}\alpha + \frac{d}{a} \end{aligned}\right\} \alpha = \frac{-b}{3a}.$$

Thus it sufficies to solve

$$y^3 + py + q = 0$$

the "depressed" cubic.

If y is a solution then $x = y - \frac{b}{3a}$ is a solution of the original.

TRICK 2: Let $y = u + v$ and try to find u, v.

$$(u+v)^3 + p(u+v) + q = 0.$$

$$u^3 + 3u^2v + 3uv^2 + v^3 + p(u+v) + q = 0$$

$$u^3 + v^3 + 3uv(u+v) + p(u+v) + q = 0$$

$$u^3 + v^3 + \overset{0}{\cancel{(3uv + p)}}(u+v) + q = 0.$$

Suppose $3uv + p = 0$. Then

$uv = -p/3$.

$$u^3 + v^3 + q = 0 \quad \text{and} \quad 3uv + p = 0.$$

$$u^3 + v^3 = -q$$

Can we find such u, v? Yes!

Note that

$$(z-u^3)(z-v^3) = z^2 - (u^3+v^3)z + u^3v^3$$

$$= \underbrace{z^2 + qz - \frac{p^3}{27}}_{\text{has roots } u^3 \,\&\, v^3.}$$

Hence

$$u^3, v^3 = \frac{-q \pm \sqrt{q^2 + 4p^3/27}}{2}$$

FINALLY,

$$y = u + v =$$

$$\sqrt[3]{\frac{-q + \sqrt{q^2 + 4p^3/27}}{2}} + \sqrt[3]{\frac{-q - \sqrt{q^2 + 4p^3/27}}{2}}$$

"Cardano's Formula"

Yikes!

$$\sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

Does it even work?

Try $x^3 - 1 = x^3 + \underset{p}{0}\,x \underset{q}{-1} = 0$

$$x = \sqrt[3]{\frac{1+\sqrt{(-1)^2+0}}{2}} + \sqrt[3]{\frac{1-\sqrt{(-1)^2+0}}{2}}$$

$$= \sqrt[3]{\frac{1+\sqrt{1}}{2}} + \sqrt[3]{\frac{1-\sqrt{1}}{2}} = \sqrt[3]{1} + \sqrt[3]{0}$$

$$= 1 + 0 = 1$$

OK

Another eg. $x^3 + 6x - 20 = 0$
has solution

$$x = \sqrt[3]{10 + \sqrt{100+8}} + \sqrt[3]{10 + \sqrt{108}}$$

$$= \sqrt[3]{10+\sqrt{108}} + \sqrt[3]{10-\sqrt{108}} \quad ?$$

$$\approx 2$$

Note $2^3 + 6 \cdot 2 - 20$
$= 8 + 12 - 20 = 0$

So $x = 2$ is a root

HW 1 due NOW.

Today: HW 1 discussion.

What does it mean for polynomials (say in $\mathbb{R}[x]$) to be "equal"?

We think of $f(x) \in \mathbb{R}[x]$ as a FUNCTION $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto f(x)$.

~~So~~

So... $f(x)$ "$=$" $g(x)$
means f and g are equal as functions

i.e. $\forall$ (for all) $\alpha \in \mathbb{R}$ we have
$f(\alpha) = g(\alpha)$.

e.g. suppose $f(x) = ax^2 + bx + c$
$g(x) = dx^2 + ex + f$.

$ax^2 + bx + c$ "$=$" $dx^2 + ex + f$.
for all values $x \in \mathbb{R}$

Put $x = 0$ to get
$c = f$

Hence $ax^2 + bx$ "=" $dx^2 + ex$.
for all x,

Differentiate

$\Rightarrow \quad 2ax + b = 2dx + e \quad \forall x.$

Put $x = 0$. $\Rightarrow \quad b = e$,

$\Rightarrow \quad 2ax = 2dx \quad \forall x.$

Diff. $\Rightarrow \quad 2a = 2d$

$\Rightarrow \quad a = d$.

Conclusion

$ax^2 + bx + c$ "=" $dx^2 + ex + f \iff$ $a = d$, $b = e$, and $c = f$.

~~as polynomials~~

~~For possibility~~

Theorem: Given $f(x), g(x) \in \mathbb{R}[x]$, we have $f(x) = g(x) \iff$ they have the same coefficients.

$$\overset{f(x)}{\overbrace{ax^2 + bx + c}} = 0$$

Now suppose $ax^2 + bx + c = 0$ has roots r & s.

Factor Theorem $\Rightarrow$ $f(x) = (x-r)\,g(x)$ where $g(x)$ has degree 1.

Now s is a root of $g(x)$. Hence $g(x) = (x-s)\,h(x)$ where $h(x)$ has degree 0

i.e. $h(x) = \underbrace{k \in \mathbb{R}}_{\text{constant}}$

Thus $ax^2 + bx + c = k(x-r)(x-s)$

$$ax^2 + bx + c \underset{\uparrow}{=} kx^2 - k(r+s)x + krs.$$

$$\begin{aligned} \Rightarrow\ & a = k \\ & b = -k(r+s) \Rightarrow r+s = -b/a \\ & c = krs \Rightarrow rs = c/a. \end{aligned}$$

We did <u>not</u> use Quad. formula.

Theorem: Suppose $f(x)$ has deg n and n roots $r_1, r_2, \dots, r_n$. Then

$$f(x) = \underset{\substack{\uparrow \\ \text{const.}}}{k} \underbrace{(x-r_1)(x-r_2)\cdots(x-r_n)}_{\text{the \underline{unique} poly of deg } n \text{ with these roots.}}$$

Let r, s be roots of $x^2 + px + q$
so $r+s = -p$ and $q = rs$.

What is "the" quad. poly with roots $\frac{1}{r}, \frac{1}{s}$?

Answer: $(x - \frac{1}{r})(x - \frac{1}{s})$

$$= x^2 - \left(\frac{1}{r} + \frac{1}{s}\right)x + \frac{1}{rs}$$

$$= x^2 - \left(\frac{r+s}{rs}\right)x + \frac{1}{rs}$$

$$= x^2 + \frac{p}{q}x + \frac{1}{q}. \quad \blacksquare$$

Express $(r-s)^2$ in terms of p, q.

i.e. in terms of $(r+s)$ & rs.

TRY:

$$(r-s)^2 = r^2 - 2rs + s^2.$$

↑ how can we get r^2 from $(r+s)$ and rs?

Answer: only from $(r+s)^2$

$$(r+s)^2 = r^2 + 2rs + s^2$$

an algorithm

$$(r-s)^2 - (r+s)^2 = (r^2 - 2rs + s^2) - (r^2 + 2rs + s^2) = -4rs.$$

DONE.

$$(r-s)^2 = (r+s)^2 - 4rs = (-p)^2 - 4q$$

$$(r-s)^2 = p^2 - 4q.$$

The "discriminant"

course notes

HW 1 solutions on the web.
HW 2 (not posted) will be due Fri Feb 11.
Exam 1: Fri Feb 18.

Where were we?

Recall: The depressed cubic $x^3+px+q=0$ has a root

$$x = \sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}} - \sqrt[3]{+\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}.$$

"Cardano's Formula" (1545)

eg. $x^3+6x-20=0$ has an "obivous" solution $x=2$.

$$\begin{array}{r|l}
 & x^2+2x+10 \\ \hline
x-2 & x^3+0x^2+6x-20 \\
 & \underline{x^3-2x^2} \\
 & 2x^2+6x-20 \\
 & \underline{2x^2-4x} \\
 & 10x-20 \\
 & \underline{10x-20} \\
 & 0 \ \checkmark
\end{array}$$

$$x^3+6x-20 = (x-2)(x^2+2x+10)$$

root 2 — root 5.

$$x = \frac{-2 \pm \sqrt{4-40}}{2}$$

impossible!

But Cardano's formula gives

$$x = \sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10}$$

$$\overset{?}{=} 2$$

YES.

1572

Rafael Bombelli treated a more worrying example

Apply Cardano's formula to $x^3-15x-4=0$ ~~has root~~ to get

$$x = \sqrt[3]{2+\sqrt{-121}} - \sqrt[3]{-2+\sqrt{-121}} = \text{impossible?}$$

~~by Cardano's formula.~~

"no real roots?"

But, by inspection

$$x^3 - 15x - 4 = (x-4)(x^2+4x+1)$$

gives roots

$x = 4$ AND $x = \dfrac{-4 \pm \sqrt{16-4}}{2}$

$$= \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}.$$

3 real roots!

Conclusion:

real

$$\sqrt[3]{2+\sqrt{-121}} - \sqrt[3]{-2+\sqrt{-121}} = 4 \text{ or } -2+\sqrt{3} \text{ or } -2-\sqrt{3}.$$

"impossible"

What!?

Bombelli was BOLD. He said, maybe:

$$(a+b\sqrt{-1})^3 = 2+\sqrt{-121} = 2+11\sqrt{-1}.$$

for some real a, b.

And then ~~he~~ solved it: $(2+\sqrt{-1})^3 = 2+11\sqrt{-1}$

Check:

$$(2+\sqrt{-1})^3 = 2^3 + 3\cdot 2^2\sqrt{-1} + 3\cdot 2\cdot(\sqrt{-1})^2 + (\sqrt{-1})^3.$$

$$= 8 + 12\sqrt{-1} - 6 + \underbrace{\sqrt{-1}\sqrt{-1}\sqrt{-1}}_{-\sqrt{-1}}$$

$$= 2 + 11\sqrt{-1}.$$

Similarly $(-2+\sqrt{-1})^3 = -2 + 11\sqrt{-1}$.

Hence (?)

$$\sqrt[3]{2+\sqrt{-121}} - \sqrt[3]{-2+\sqrt{-121}}$$

$$= (2+\sqrt{-1}) - (-2+\sqrt{-1}) = 4 \quad \text{wow!}$$

"shortest path between two truths.." Hadamard.
"an amphibian ..." Leibniz.
"as subtle as they are useless" Cardano.

Conclusion: $\sqrt{-1}$ is <u>not</u> <u>useless</u>.

What <u>is</u> it?