INVESTIGATIONS ON THE IMAGINARY ROOTS OF EQUATIONS

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1. Every algebraic equation which has been freed of fractions and radical signs always reduces to this general form

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \cdots + N = 0,$$

where the letters A, B, C, D, $\ldots$, N indicate constant real quantities, either positive or negative, not excluding zero. The roots of such an equation are the values which when put for x produce an identity equation $0 = 0$. Now if $x + \alpha$ is a divisor or a factor of the given formula, the other factor being indicated by X, so that the equation has this form

$$(x + \alpha)X = 0,$$

then it is clear that this happens when

$$x + \alpha = 0, \quad \text{or} \quad x = -\alpha.$$

From this we see that the roots of an equation are found by looking for the divisors or factors of this same equation; and all the roots of an equation are derived from all the simple divisors of the form $x + \alpha$.

2. So to find all the roots of a given equation, we have only to look for all the simple factors of the quantity

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \cdots + N,$$

and if we set these factors:

$$(x + \alpha)(x + \beta)(x + \gamma)(x + \delta) \cdots$$

Recherches sur les racines imaginaires des équations.
Mémoires de l'académie des sciences de Berlin [5] (1749), 1751 p. 222–288.
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then it is immediately clear that the number of these factors must be equal to the exponent n; and therefore the number of all the roots, which will be

$$x = -\alpha, \quad x = -\beta, \quad x = -\gamma, \quad x = -\delta, \quad \dots,$$

will also equal this same exponent n, since a product such as

$$(x+\alpha)(x+\beta)(x+\gamma)(x+\delta)\cdots$$

cannot become equal to zero unless one of its factors vanishes. Every equation then, of whatever degree, will always have as many roots as the exponent of its highest power contains units.

3. Now it very often happens that not all of these roots are real quantities, and that some, or perhaps all, are imaginary quantities. We call a quantity imaginary when it is neither greater than zero, nor less than zero, nor equal to zero. This will be then something impossible, as for example $\sqrt{-1}$, or in general $a + bi$, since such a quantity is neither positive, nor negative, nor zero. So in this equation

$$x^3 - 3xx + 6x - 4 = 0$$

which has these three roots

$$x = 1, \quad x = 1 + \sqrt{-3}, \quad x = 1 - \sqrt{-3},$$

the last two are imaginary, and there is only one real root, $x = 1$. From this we see that if we wish to include under the name of roots only those which are real, their number would often be much smaller than the highest exponent in the equation. And therefore when we say that every equation has as many roots as its degree exponent indicates, that must be understood to include all the roots, both real and imaginary.

4. We imagine then, that whatever the degree of the given equation,

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \cdots + N = 0,$$

it can always be represented by a form such as

$$(x+\alpha)(x+\beta)(x+\gamma)(x+\delta)\cdots(x+\nu) = 0,$$

where the number of these simple factors is n. And since these factors when explicitly multiplied together must produce the given equation, it is evident that the quantities A, B, C, D, $\ldots$, N will be determined by the quantities α, β, γ, δ, $\ldots$, ν, in such a way that we will have

$$A = \text{the sum of these quantities } \alpha,\ \beta,\ \gamma,\ \delta,\ \ldots,\ \nu,$$

$$B = \text{the sum of all their products taken two at a time,}$$

$$C = \text{the sum of all their products taken three at a time,}$$

$$D = \text{the sum of all their products taken four at a time,}$$

$$\vdots$$

and finally

$$N = \text{the product taken all together, } \alpha\beta\gamma\delta\cdots\nu.$$

And since the number of these equalities is n, the values of the letters α, β, γ, δ, $\ldots$, ν will, conversely, be determined by them.

5. Although it seems that knowledge of imaginary roots might not have any use, considering that they do not provide any solution to whatever problem there may be, nevertheless it is very important in all of analysis to become familiar with the calculation of imaginary quantities. For not only will we acquire from it a more perfect knowledge of the nature of equations, but analysis of the infinite will derive very considerable benefit from it. For each time it comes up that we must integrate a fraction, it is necessary to resolve the denominator into all its simple factors, be they real or imaginary, and from there we finally derive the integral, which although it contains imaginary logarithms, we have the means to reduce them to arcs of real circles. Besides that, it often happens that an expression which contains imaginary quantities is nevertheless real, and in these cases calculation with imaginary quantities is absolutely necessary.

6. It is shown in algebra that when an equation has imaginary roots, their number is always even, so that every equation either has no imaginary roots at all, or else it has two of them, or else four, or six, or eight, etc., and never can the number of all the imaginary roots of an equation be odd. But we further hold that the imaginary roots pair up in such a way that both the sum and the product of the two become real. Or what comes to the same thing, if

$$x + yi$$

is one of the imaginary factors of an equation, we hold that there will be found among the others a factor

$$x - yi,$$

also imaginary, which when multiplied by the former $x + yi$ gives a real product. The product of $x + yi$ and $x - yi$ is equal to $xx + yy$, and the sum is equal to $2x$, so it is clear that both are real quantities.

7. To make this clearer, let $2m$ be the number of simple imaginary factors of an arbitrary equation, since we know that the number is even, and we hold that one can always arrange these factors in pairs so that their products become real. So these imaginary factors numbering $2m$ are reduced to real factors numbering m, and these latter factors will no longer be simple, but of the form

$$xx + px + q,$$

so they will be of the second degree. We say then that every equation which cannot be resolved into simple real factors always has real factors of the second degree. However nobody, as far as I know, has yet proved sufficiently rigorously the truth of this sentiment. I will try therefore to give it a proof which would not be subject to any exception.

8. First, it is evident that when an equation has only two simple imaginary factors, their product is necessarily real. Because the product of these two factors multiplied by the product of all the others, which we suppose to be real, must produce the given equation, that is to say a real quantity, which would be impossible if the product of the two imaginary factors were not real. We see, moreover, that if an equation has four imaginary roots, all the others being real, the product of these four imaginary factors will also be real. And in general whatever the number of imaginary factors of an equation, their product must necessarily be a real quantity; so if the number of imaginary factors of an equation is equal to $2m$, the product of all these factors multiplied together will be of this form

$$x^{2m} + ax^{2m-1} + bx^{2m-2} + cx^{2m-3} + \cdots,$$

where all the coefficients a, b, c, etc. are real quantities.

9. It is necessary therefore to begin by proving that an equation of the fourth degree

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

of which all the roots are imaginary, is always resolvable into two real factors of the second degree

$$(xx + px + r)(xx + qx + s) = 0,$$

because if all the roots are real, or two at least, such a resolution will not present any difficulty. But if all four are imaginary, the thing is not only less evident, but there are even cases which would not seem to admit such a resolution. A very wise geometer once suggested to me this equation

$$x^4 + 2x^3 + 4x^2 + 2x + 1 = 0$$

by which he wished to prove that the resolution into two real factors was not always possible. And indeed it seems at first very difficult to combine these four simple imaginary factors pairwise so that their products become real.

10. The doubt raised by this equation is too important for me to skip over it by giving a general proof of the matter in question, so I will carefully develop this case before undertaking this proof. So first, since the coefficients of this equation are 1, 2, 4, 2, 1, and take the same order starting from the front or the back, it is certain that the given equation is resolvable into two factors of this form

$$xx + px + 1, \quad xx + qx + 1,$$

whose product

$$x^4 + (p+q)x^3 + (pq+2)x^2 + (p+q)x + 1,$$

when compared with the given form

$$x^4 + 2x^3 + 4x^2 + 2x + 1$$

furnishes these two equalities

$$\begin{aligned} p + q &= 2 \\ pq + 2 &= 4, \quad \text{or} \quad pq = 2. \end{aligned}$$

Then we will have

$$(p-q)^2 = (p+q)^2 - 4pq = 4 - 4 \cdot 2 = -4$$

and therefore

$$p - q = \sqrt{-4} = 2i$$

from which we derive these values

$$p = 1 + i \quad \text{and} \quad q = 1 - i,$$

so that the given equation

$$x^4 + 2x^3 + 4x^2 + 2x + 1 = 0$$

is now reduced to these two factors of the second degree

$$(xx + (1+i)x + 1)\,(xx + (1-i)x + 1) = 0,$$

which are in truth imaginaries.

11. But to decide whether or not it is possible to reduce this equation to two real factors of the second degree, it is necessary to look to its four simple factors, in order to see whether we can recombine them in pairs, so as to get two real products. The first double factor, $xx + (1+i)x + 1$, gives these two simple factors

$$x + \tfrac{1}{2}(1+i) + \tfrac{1}{2}\sqrt{2i-4} = 0,$$
$$x + \tfrac{1}{2}(1+i) - \tfrac{1}{2}\sqrt{2i-4} = 0,$$

and the other double factor, $xx + (1-i)x + 1$, gives these two simple factors

$$x + \tfrac{1}{2}(1-i) + \tfrac{1}{2}\sqrt{-2i-4} = 0,$$
$$x + \tfrac{1}{2}(1-i) - \tfrac{1}{2}\sqrt{-2i-4} = 0.$$

It is a question then, of seeing whether or not the first simple factor multiplied by the third or the fourth produces a double real factor, since we already see that the product of the first and the second is imaginary.

12. However it is not so easy to recognize whether the products we get by these multiplications of the first factor by the third or by the fourth are real or imaginary, and the difficulty arises from the imaginary terms $\sqrt{2i-4}$ and $\sqrt{-2i-4}$, the imaginary parts of which we cannot compare with those of the other numbers $1+i$ and $1-i$. I note that the formula $\sqrt{2i-4}$ can be reduced to the form $u+vi$, and then the other formula $\sqrt{-2i-4}$ will become equal to $u-vi$. For making these equalities

$$\sqrt{2i-4} = u+vi$$

and

$$\sqrt{-2i-4} = u-vi$$

and taking the squares, we will obtain these

$$2i-4 = uu-vv+2uvi$$

and

$$-2i-4 = uu-vv-2uvi,$$

from which we will derive

$$-4 = uu-vv \quad \text{and} \quad 2i = 2uvi,$$

or

$$vv-uu = 4 \quad \text{and} \quad uv = 1.$$

And then we will form

$$(vv+uu)^2 = (vv-uu)^2 + 4uuvv = 16+4 = 20,$$

so that

$$vv+uu = \sqrt{20} = 2\sqrt{5}.$$

From that we will at last find

$$vv = \sqrt{5}+2 \quad \text{and} \quad uu = \sqrt{5}-2$$

and consequently

$$v = \sqrt{\sqrt{5}+2} \quad \text{and} \quad u = \sqrt{\sqrt{5}-2}.$$

13. These two values v and u being real, let us substitute them into the expressions for the four simple factors found above, and these factors will become

$$\begin{aligned}
\text{I.} &\quad x+\tfrac{1}{2}(1+i)+\tfrac{1}{2}(u+vi) = x+\tfrac{1}{2}(1+u)+\tfrac{1}{2}(1+v)i,\\
\text{II.} &\quad x+\tfrac{1}{2}(1+i)-\tfrac{1}{2}(u+vi) = x+\tfrac{1}{2}(1-u)+\tfrac{1}{2}(1-v)i,\\
\text{III.} &\quad x+\tfrac{1}{2}(1-i)+\tfrac{1}{2}(u-vi) = x+\tfrac{1}{2}(1+u)-\tfrac{1}{2}(1+v)i,\\
\text{IV.} &\quad x+\tfrac{1}{2}(1-i)-\tfrac{1}{2}(u-vi) = x+\tfrac{1}{2}(1-u)-\tfrac{1}{2}(1-v)i.
\end{aligned}$$

It is now clear that the product of the first by the third actually becomes real, as well as that of the second by the fourth. For we will have

$$\begin{aligned}\text{product of I by III } &= \big(x + \tfrac{1}{2}(1+u)\big)^2 + \tfrac{1}{4}(1+v)^2,\\ \text{product of II by IV } &= \big(x + \tfrac{1}{2}(1-u)\big)^2 + \tfrac{1}{4}(1-v)^2.\end{aligned}$$

We see then that the given equation

$$x^4 + 2x^3 + 4x^2 + 2x + 1 = 0$$

reduces to these two real factors of the second degree:

$$\begin{aligned}xx + (1+u)x + \tfrac{1}{2} + \tfrac{1}{2}(u+v) + \tfrac{1}{4}(vv+uu) = 0,\\ xx + (1-u)x + \tfrac{1}{2} - \tfrac{1}{2}(u+v) + \tfrac{1}{4}(vv+uu) = 0,\end{aligned}$$

where

$$v = \sqrt{\sqrt{5}+2}, \quad u = \sqrt{\sqrt{5}-2}$$

and

$$vv + uu = 2\sqrt{5}.$$

14. This example leads us to a more general problem, which will not fail to considerably enlighten us on the subject in question.

This problem considers this more general fourth-degree equation

$$x^4 + ax^3 + (b+2)xx + ax + 1 = 0,$$

which we must resolve into two double factors, or factors of the second degree, which must be real.

Let us first set these two factors in the following form

$$(xx + px + 1)(xx + qx + 1) = 0$$

and we see first that it is necessary that

$$p + q = a \quad \text{and} \quad pq = b,$$

from which we will derive

$$p = \frac{a + \sqrt{aa - 4b}}{2} \quad \text{and} \quad q = \frac{a - \sqrt{aa - 4b}}{2}.$$

So whenever $aa > 4b$ the problem is solved, considering that the two assumed factors become real.

15. But when $aa < 4b$, these two factors will be imaginary, and will not satisfy the question. In this case it is necessary to consider the simple factors, which will be

$$\text{I.}\quad x + \tfrac{1}{2}p + \tfrac{1}{2}\sqrt{pp-4} = 0,$$

$$\text{II.}\quad x + \tfrac{1}{2}p - \tfrac{1}{2}\sqrt{pp-4} = 0,$$

$$\text{III.}\quad x + \tfrac{1}{2}q + \tfrac{1}{2}\sqrt{qq-4} = 0,$$

$$\text{IV.}\quad x + \tfrac{1}{2}q - \tfrac{1}{2}\sqrt{qq-4} = 0.$$

Let us set $4b = aa + cc$, since $aa < 4b$, and we will have

$$p = \frac{a+ci}{2} \quad \text{and} \quad q = \frac{a-ci}{2},$$

so

$$\sqrt{pp-4} = \tfrac{1}{2}\sqrt{aa-cc-16+2aci}$$

and

$$\sqrt{qq-4} = \tfrac{1}{2}\sqrt{aa-cc-16-2aci}.$$

Since these two formulas are imaginary, let

$$\sqrt{aa-cc-16+2aci} = u+vi$$

and

$$\sqrt{aa-cc-16-2aci} = u-vi$$

and from this we will derive

$$uu-vv = aa-cc-16 \quad \text{and} \quad uv = ac.$$

16. These equalities give us

$$(uu+vv)^2 = (aa-cc-16)^2 + 4aacc = (aa+cc)^2 - 32(aa-cc) + 256,$$

so

$$vv+uu = \sqrt{(aa+cc)^2 - 32(aa-cc) + 256}.$$

Since this irrational quantity has the sum of two squares under the radical sign, it will always be real, and its value will even be greater than both

$$aa - cc - 16 = uu - vv \quad \text{and} \quad vv - uu = 16 + cc - aa.$$

We will therefore have the following real values for v and u:

$$v = \sqrt{\frac{\sqrt{(aa+cc)^2 - 32(aa-cc) + 256} + 16 + cc - aa}{2}},$$

$$u = \sqrt{\frac{\sqrt{(aa+cc)^2 - 32(aa-cc) + 256} - 16 - cc + aa}{2}},$$

and substituting for cc its value $4b - aa$, we will have

$$v = \sqrt{\sqrt{4bb - 16(aa - 2b) + 64} + 8 + 2b - aa},$$

$$u = \sqrt{\sqrt{4bb - 16(aa - 2b) + 64} - 8 - 2b + aa},$$

or

$$v = \sqrt{2\sqrt{(b+4)^2 - 4aa} + 8 + 2b - aa},$$

$$u = \sqrt{2\sqrt{(b+4)^2 - 4aa} - 8 - 2b + aa}.$$

17. Having found these real values for v and u in the case where $4b > aa$ or $4b = aa + cc$, our four simple imaginary factors will be

$$\text{I.} \quad x + \tfrac{1}{4}(a + ci) + \tfrac{1}{4}(u + vi) = x + \tfrac{1}{4}(a + u) + \tfrac{1}{4}(c + v)i,$$

$$\text{II.} \quad x + \tfrac{1}{4}(a + ci) - \tfrac{1}{4}(u + vi) = x + \tfrac{1}{4}(a - u) + \tfrac{1}{4}(c - v)i,$$

$$\text{III.} \quad x + \tfrac{1}{4}(a - ci) + \tfrac{1}{4}(u - vi) = x + \tfrac{1}{4}(a + u) - \tfrac{1}{4}(c + v)i,$$

$$\text{IV.} \quad x + \tfrac{1}{4}(a - ci) - \tfrac{1}{4}(u - vi) = x + \tfrac{1}{4}(a - u) - \tfrac{1}{4}(c - v)i,$$

from which it is clear that the products of factors I by III, and of II by IV, are real, becoming

$$xx + \tfrac{1}{2}(a + u)x + \tfrac{1}{16}(aa + cc) + \tfrac{1}{16}(uu + vv) + \tfrac{1}{8}(au + cv),$$

$$xx + \tfrac{1}{2}(a - u)x + \tfrac{1}{16}(aa + cc) + \tfrac{1}{16}(uu + vv) - \tfrac{1}{8}(au + cv),$$

where it must be noted that

$$aa + cc = 4b$$

and

$$vv + uu = 4\sqrt{(b+4)^2 - 4aa}.$$

18. To express more conveniently the value of $au + cv$, let us look to the square, $aauu + ccvv + 2acuv$:

$$aauu = 2aa\sqrt{(b+4)^2 - 4aa} - 8aa - 2aab + a^4,$$

$$ccvv = (8b - 2aa)\sqrt{(b+4)^2 - 4aa} + 32b + 8bb - 4aab - 8aa - 2aab + a^4,$$

$$2acuv = 2aacc = 8aab - 2a^4,$$

so we will have

$$(au + cv)^2 = 8b\sqrt{(b+4)^2 - 4aa} + 32b - 16aa + 8bb$$

and the square root will be found to be

$$au + cv = 2\sqrt{2bb + 8b - 4aa + 2b\sqrt{(b+4)^2 - 4aa}}$$

and therefore the two real factors sought will be, in the case of $4b > aa$,

$$\begin{aligned} xx + \tfrac{1}{2}ax + \tfrac{1}{2}x\sqrt{2\sqrt{(b+4)^2 - 4aa} - 8 - 2b + aa} \\ + \tfrac{1}{4}b + \tfrac{1}{4}\sqrt{(b+4)^2 - 4aa} + \tfrac{1}{4}\sqrt{2bb + 8b - 4aa + 2b\sqrt{(b+4)^2 - 4aa}}, \end{aligned}$$

and

$$\begin{aligned} xx + \tfrac{1}{2}ax - \tfrac{1}{2}x\sqrt{2\sqrt{(b+4)^2 - 4aa} - 8 - 2b + aa} \\ + \tfrac{1}{4}b + \tfrac{1}{4}\sqrt{(b+4)^2 - 4aa} - \tfrac{1}{4}\sqrt{2bb + 8b - 4aa + 2b\sqrt{(b+4)^2 - 4aa}}. \end{aligned}$$

19. By this particular case one will more easily understand what I want to prove in general: that an equation of arbitrary degree is always resolvable into real factors, either simple or double. Or since two simple factors joined together produce a double factor, we must show that every equation of even degree, as

$$x^{2m} + Ax^{2m-1} + Bx^{2m-2} + \cdots + N = 0,$$

is resolvable into m real double factors of the form $xx + px + r$, and that an equation of odd degree, as

$$x^{2m+1} + Ax^{2m} + Bx^{2m-1} + \cdots + N = 0,$$

has first one simple real factor, and then m double factors which are also all real. For this purpose I will develop the following propositions, which will lead to the proof of what I just advanced.

Theorem 1.

20. *Every equation of odd degree, whose general form is*

$$x^{2m+1} + Ax^{2m} + Bx^{2m-1} + Cx^{2m-2} + \cdots + N = 0,$$

always has at least one real root, and if it has more than one, their number will be odd.

Proof

Let us set

$$x^{2m+1} + Ax^{2m} + Bx^{2m-1} + \cdots + N = y$$

and consider the curve expressed by this equation. It is evident that each abscissa x corresponds to only a single ordinate y, which will always be real, and that at the place where the ordinate y vanishes, the value of the abscissa x will be a root of the given equation. So this equation will have as many real roots as there are places where the ordinate y vanishes, which happens where the curve crosses the axis of the abscissas. And so the number of real roots will be equal to the number of intersections of the curve with the axis on which we take the abscissas. To judge then the number of these intersections, let us first set the abscissa x positive and infinitely large, or $x = \infty$, and it is clear that it will then become

$$y = \infty^{2m+1} = \infty,$$

from which it follows that the branch of the curve corresponding to the infinitely positive abscissas is found above the axis, since their ordinates y

are positive. Now setting the negative and also infinite abscissas, or $x = -\infty$, we will have

$$y = (-\infty)^{2m+1} = -\infty,$$

so the ordinates here will be negative, and the branch of the curve will be found below the axis. This branch is continuous with the other one situated above the axis, so it is absolutely necessary that the curve cross some part of the axis, and if it crosses at several points, the number of these points must be odd. From this it follows that the given equation will necessarily have at least one real root, and if it has more, their number will always be odd. Q.E.D.

Corollary

21. Therefore, since the number of all the roots of the given equation is equal to $2m + 1$, or odd, and the number of real roots is also odd, it follows that the number of imaginary roots, if there are any, will always be even.

Theorem 2.

22. *Every equation of even degree, whose general form is*

$$x^{2m} + Ax^{2m-1} + Bx^{2m-2} + \cdots + N = 0,$$

either has no real roots at all, or if it has real roots, their number will always be even.

Proof

Let us consider again the curve expressed by this equation

$$x^{2m} + Ax^{2m-1} + Bx^{2m-2} + \cdots + N = y,$$

which consists of only a single continuous line, since at each abscissa x there always corresponds a single ordinate. Let us set $x = +\infty$, and we will again have

$$y = +\infty,$$

so the branch of the curve corresponding to the infinitely positive abscissas will be situated above the axis. Now setting $x = -\infty$, we will similarly have

$$y = (-\infty)^{2m} = +\infty,$$

so that the branch of the curve corresponding to the infinitely negative abscissas will also be found above the axis. Therefore it will be possible that the curve does not cross any part of the axis of abscissas; and if it does pass over some part of the axis in order to descend into the region below, it must pass over it again in order to return to the region above. Consequently, if the curve crosses the axis, it must be that the number of all the intersections is even. Therefore, since each intersection gives a real root of the given equation, it follows that either it will not have any real roots at all, or if it has some, their number will always be even. Q.E.D.

Corollary

23. Since the number of all the roots, both real and imaginary, of the given equation is $2m$, and therefore even, and since the number of the real roots, if it has any, is also even, it follows that the number of the imaginary roots, if it has any, is also even.

Scholium

24. These two theorems with their proofs are already so well known that I would have been able to report them here without explaining them in detail. But since they involve the foundation of the whole theory—the number of all the imaginary roots of an arbitrary equation is always even—I believed it necessary to derive them from the beginning, and this all the more so because the following theorem, which is not as generally known, requires a similar proof.

Theorem 3.

25. Every equation of even degree where the last term, or constant, has a negative value, as

$$x^{2m} + Ax^{2m-1} + Bx^{2m-2} + \cdots - OO = 0,$$

always has at least two real roots, one positive and the other negative.

Proof

Set

$$x^{2m} + Ax^{2m-1} + Bx^{2m-2} + \cdots - OO = y,$$

in order to consider the curve expressed by this equation. We have just seen that this curve extends on both sides to infinity above the axis. Now by setting $x = 0$ we will have

$$y = -OO,$$

and therefore the point of the curve which corresponds to $x = 0$ will be below the axis, so it is necessary that the curve cross the axis on both sides of this point, in order to rise above. So since each intersection gives a real root of the given equation, and since one of these two intersections must correspond to a positive abscissa x and the other a negative, it is certain that the given equation will have at least two real roots, one positive and the other negative. Q.E.D.

Corollary

26. This proof makes us also understand that when an equation like the given one has several real positive roots, their number will be odd, and the number of negative real roots will be odd as well.

Theorem 4.

27. *Every fourth-degree equation, as*

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0,$$

can always be decomposed into two real factors of the second degree.

Proof

We know that setting $x = y - \frac{1}{4}A$ will change this equation into another of the same degree, where the second term is absent; and because this transformation can always be done, let us suppose that in the given equation the second term is already absent, and that we will have this equation

$$x^4 + Bx^2 + Cx + D = 0$$

to resolve into to real factors of the second degree. It is first clear that these two factors will be of this form

$$(xx + ux + \alpha)(xx - ux + \beta).$$

Comparing the product with the given equation, we will have

$$B = \alpha + \beta - uu, \quad C = (\beta - \alpha)u, \quad D = \alpha\beta,$$

from which we will derive

$$\alpha + \beta = B + uu, \quad \beta - \alpha = \frac{C}{u},$$

and therefore

$$2\beta = uu + B + \frac{C}{u} \quad \text{and} \quad 2\alpha = uu + B - \frac{C}{u}.$$

Then having $4\alpha\beta = 4D$, we will obtain this equation

$$u^4 + 2Buu + BB - \frac{CC}{uu} = 4D$$

or

$$u^6 + 2Bu^4 + (BB - 4D)uu - CC = 0,$$

from which it is necessary to look for the value of u. Since the constant term $-CC$ is strictly negative, we just proved that this equation has at least two real roots. So taking one or the other for u, the values α and β will be likewise real, and consequently the two assumed factors of the second degree, $xx + ux + \alpha$ and $xx - ux + \beta$, will be real. Q.E.D.

Corollary 1.

28. Every expression then of the fourth degree,

$$x^4 + Ax^3 + Bx^2 + Cx + D,$$

although all its four simple factors may be imaginary, can always be decomposed into two real factors of the second degree. Or indeed each of the four simple factors has among the others its companion, and when multiplied by it, it produces a real product.

Corollary 2.

29. And, if an expression of arbitrary degree has only four simple imaginary factors, since their product is real and of the form $x^4 + Ax^3 + Bx^2 + Cx + D$, it is also certain that this product is resolvable into two real factors of the second degree, each of which contains two simple imaginary factors.

Corollary 3.

30. From this it is also evident that an arbitrary equation of the fifth degree is always resolvable into three real factors, one of which is simple and the two others double, or of second degree. For this equation has a real root, and so will have a simple real factor, and the other factor will be of the fourth degree, and so can be decomposed into two double real factors.

Corollary 4.

31. The resolution of equations into real factors, either simple or double, is therefore proved for equations of the fifth degree and for all the lesser degrees. But this theorem is not sufficient to prove this resolution for any greater degree, unless the number of imaginary roots is smaller than 6. For then the number would be either 4, or 2, or 0, and in all these cases it is evident that this resolution is possible.

Scholium 1.

32. I already proved above that this equation of the fourth degree

$$x^4 + ax^3 + (b+2)x^2 + ax + 1 = 0,$$

which is only a special case of the general for this degree, which I just considered here, is always resolvable into two real factors of the second degree. This resolution, which was quite troublesome in the case of $4b > aa$, can be immediately deduced by the method used in this theorem, without regard to the form of the imaginary roots. This use seems to me important enough that I apply the general resolution to this case. To avoid fractions let us set $a = 4c$ and $b > 4cc$, so that the equation to resolve is

$$x^4 + 4cx^3 + (b+2)xx + 4cx + 1 = 0.$$

Now, to remove the second term set $x = y - c$ and our equation will take this form

$$y^4 + (2 + b - 6cc)y^2 + (8c^2 - 2bc)y + 1 - 2cc + bcc - 3c^4 = 0.$$

We suppose the real factors of the second degree to be

$$(yy + uy + \alpha)(yy - uy + \beta) = 0,$$

and because

$$B = 2 + b - 6cc, \quad C = 8c^2 - 2bc, \quad \text{and} \quad D = 1 - 2cc + bcc - 3c^4,$$

in order to find u we will have this equation to resolve

$$u^6 + (4 + 2b - 12cc)u^4 + (bb + 4b - 16bcc - 16cc + 48c^4)u^2 - 4cc(4cc - b)^2 = 0,$$

which when divided by $uu + b - 4c$ gives

$$u^4 + (4 + b - 8cc)u^2 + 16c^4 - 4bcc = 0.$$

The first factor, $uu + b - 4cc$, when set to zero gives only imaginary values for u, because $b > 4cc$. So it is necessary to look to the other equation for a real value, and we derive

$$uu = -2 - \tfrac{1}{2}b + 4cc \pm \sqrt{(2 + \tfrac{1}{2}b)^2 - 16cc},$$

and the real value of u will be

$$u = \sqrt{\sqrt{(2 + \tfrac{1}{2}b)^2 - 16cc} - 2 - \tfrac{1}{2}b + 4cc}$$

or putting aa for $16cc$

$$u = \tfrac{1}{2}\sqrt{2\sqrt{(b + 4)^2 - 4aa} - 8 - 2b + aa},$$

from which we find the same factors that were attributed above.

Scholium 2.

33. The force of the proof of this theorem comes down to the unknown u being determined by an equation of the sixth degree, and the last term of this equation being strictly negative. Both of these two characteristics can be discovered through pure reasoning, without even having to look for the equation that contains the unknown u. Therefore, since in what follows, where I will go on to equations of higher degree, it would be very difficult and even impossible to find the equation by which the unknown u is determined, it will be important to discover these two mentioned characteristics through pure reasoning for the given equation of the fourth degree, in order to clear the way for using this same reasoning when the proposed equation will be of a higher degree.

So let the given equation be already cleared of the second term

$$x^4 + Bx^2 + Cx + D = 0,$$

and set the four roots of this equation to

$$x = \mathfrak{a}, \quad x = \mathfrak{b}, \quad x = \mathfrak{c}, \quad x = \mathfrak{d},$$

and it is clear first that the sum of these four roots

$$\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d}$$

will be equal to zero.

Next, setting one of the general double factors of this equation to zero

$$xx - ux - \beta = 0$$

it is certain that u will be the sum of two arbitrary roots out of the four specified $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$. Therefore this letter u regarded as our unknown can have as many different values as there are distinct combinations of two letters taken from these four $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$. This number of combinations is, as we know, equal to $\frac{4\cdot 3}{1\cdot 2} = 6$, and so the letter u is susceptible to 6 different values, and no more. Therefore, the letter u will be determined by an equation of the sixth degree, which will have the following six roots

$$\begin{array}{rlrlrl} \text{I.} & u = \mathfrak{a} + \mathfrak{b}, & \text{II.} & u = \mathfrak{a} + \mathfrak{c}, & \text{III.} & u = \mathfrak{a} + \mathfrak{d}, \\ \text{IV.} & u = \mathfrak{c} + \mathfrak{d}, & \text{V.} & u = \mathfrak{b} + \mathfrak{d}, & \text{VI.} & u = \mathfrak{b} + \mathfrak{c}. \end{array}$$

So since $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = 0$, if we set the first three of these six roots

$$\text{I.}\quad u = p, \qquad \text{II.}\quad u = q, \qquad \text{III.}\quad u = r,$$

then the last three will be

$$\text{IV.}\quad u = -p, \qquad \text{V.}\quad u = -q, \qquad \text{VI.}\quad u = -r,$$

so that the negative of each value of u will also be a value of u.

Knowing now these six roots, the equation which furnishes them all will be

$$(u - p)(u - q)(u - r)(u + p)(u + q)(u + r) = 0,$$

or by combining pairs where one is the negative of the other, we will have

$$(uu - pp)(uu - qq)(uu - rr) = 0,$$

which will give an equation of the sixth degree where all the odd powers are absent, just as we found in the proof of this theorem.

But I observe, furthermore, that the last constant term of this equation will be equal to

$$-pp \cdot -qq \cdot -rr = -ppqqrr,$$

which is then a square with the negative sign, so it is strictly negative. From this it follows that this equation will necessarily have at least two real roots, and one or the other taken for u will give a real double factor of the given equation. We see, then, another proof of the given theorem, which will be similar to those of the theorems which follow.

One will no doubt object that I have assumed here that the quantity pqr is a real quantity, and that its square, $ppqqrr$, is positive, and that this is still uncertain, considering that the roots $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ are imaginary, and it might well happen that the square of the quantity pqr, being composed of them, is negative. To this I respond that this case can never occur, because however imaginary be the roots $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, we nevertheless know that we must have

$$\begin{gathered}
\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}=0,\\
\mathfrak{ab}+\mathfrak{ac}+\mathfrak{ad}+\mathfrak{bc}+\mathfrak{bd}+\mathfrak{cd}=B,\\
\mathfrak{abc}+\mathfrak{abd}+\mathfrak{acd}+\mathfrak{bcd}=-C,\\
\mathfrak{abcd}=D,
\end{gathered}$$

these quantities B, C, D, being real. But since

$$\begin{gathered}
p=\mathfrak{a}+\mathfrak{b},\\
q=\mathfrak{a}+\mathfrak{c},\\
r=\mathfrak{a}+\mathfrak{d},
\end{gathered}$$

their product

$$pqr=(\mathfrak{a}+\mathfrak{b})(\mathfrak{a}+\mathfrak{c})(\mathfrak{a}+\mathfrak{d})$$

is determinable, as we know, by the quantities B, C, D, and will consequently be real. Just as we have seen, it is actually

$$pqr=-C \quad \text{and} \quad ppqqrr=CC$$

We will easily recognize as well that in the higher equations this same characteristic must occur, and that one could not object on this basis to the following theorems.

Theorem 5.

34. *Every equation of degree 8 is always resolvable into two real factors of the fourth degree.*

Proof

After eliminating the second term, the given degree 8 equation will have this form

$$x^8 + Bx^6 + Cx^5 + Dx^4 + Ex^3 + Fx^2 + Gx + H = 0,$$

of which the two general fourth-degree factors will be

$$x^4 - ux^3 + \alpha x^2 + \beta x + \gamma = 0,$$
$$x^4 + ux^3 + \delta x^2 + \epsilon x + \zeta = 0.$$

If we equate the product of these two factors to the given equation, we will obtain seven equalities, which is to say precisely as many as there are unknown coefficients u, α, β, γ, δ, ϵ, ζ. From these equalities we will successively eliminate the letters α, β, γ, δ, ϵ, ζ, which can always be done, as we know, without having to extract any roots, so that the values of these letters will all be real expressions of the known quantities B, C, D, E, F, G, H, and the unknown u, and so finally we will reach one equation that will contain only the unknown u along with the known quantities, from which we must find the value of u. And this value having been found to be real, the values of the eliminated letters α, β, γ, etc. will also be real, and therefore the two assumed fourth-degree factors likewise real.

It is a question, then, of finding the equation which determines for us the value of u. In general u will express the sum of four arbitrary roots of the given equation, of which the number of all the roots is equal to eight, and so the letter u will have as many different values as there are various combinations of four roots taken from the eight of the equation. Thus the number of all the values of u will be equal to $\frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4} = 70$, and therefore the unknown u will be determined by an equation of degree 70. Moreover, if we suppose that p is one of the values of u, p will be the sum of some four roots of the given equation, and the sum of the other four will be equal to $-p$, since the sum of all eight roots is equal to zero. Thus if $u - p$ is a factor of the degree 70 equation, $u+p$ will also be one, and therefore joining these two factors together, $uu - pp$ will be a double factor, or a second degree factor of the above-mentioned equation of degree 70. Consequently, this equation will have 35 factors of the form $uu - pp$, or it will be a product such as

$$(uu - pp)(uu - qq)(uu - rr)(uu - ss)\cdots$$

the number of these factors being equal to 35. So the last or constant term of this equation will be the product of 35 negative squares, and consequently also a negative square like $-ppqqrrss$ etc., because the number 35 is odd. The root of this square, $pqrs$ etc., is a real quantity determinable by the coefficients B, C, D, E, etc. of the given equation, and therefore its square, $ppqqrrss$ etc., a positive quantity. So the unknown coefficient u being determined by an equation of degree 70 whose last term is strictly negative, this equation will have at least two real values, of which one being put for u will furnish a real factor of the fourth degree of the given equation, which will consequently be resolvable into two real factors of the fourth degree. Q.E.D.

Corollary 1.

35. Since each factor of the fourth degree is resolvable into two real factors of the second degree, it follows that every equation of degree 8 is always resolvable into four real factors of the second degree of the form $xx + px + q$.

Corollary 2.

36. We also see that every equation of the ninth degree is resolvable into a simple real factor and four double or second-degree factors, also real.

Corollary 3.

37. This proposition makes us also see that the same resolution into real factors, either simple or double, must also occur in all equations of the sixth or seventh degree. For we only have to multiply such an equation either by x or by xx, to reduce it to degree 8.

Scholium 1.

38. Having multiplied an equation of the sixth degree by xx, in order to get one of degree 8, the two fourth-degree factors of the latter will contain this multiplier xx, which we will consequently have to remove in order to get the factors of the given sixth-degree equation. It will happen either that one of the two fourth-degree factors will contain xx, or else that each of them will contain x. In the first case we will have after division by xx a real second-degree factor and one of the fourth, which being separated by two from the second, we will have the three double factors of the given equation. And in the second case, dividing each factor by x, we will obtain two real third-degree factors, each containing a simple real factor, so that in either case the equation of the sixth degree is resolved into real factors, either simple or double. We will see furthermore that the seventh-degree equations are likewise resolvable into such factors, since we know that these equations always have a simple real factor, dividing by which they will be reduced to equations of the sixth degree.

Scholium 2.

39. It seems still uncertain whether after having found a real value for u, the other coefficients α, β, γ, δ, etc. will also be determined by a real expression, seeing that it could happen that some might contain irrational quantities, which might become imaginary. But to lift this doubt, we only have to regard u as an already-known quantity, so that the number of the equalities to satisfy surpasses by one the number of the unknowns α, β, γ, δ, etc. which we are to determine. So we will eliminate one of these quantities after another, as this can be done without extracting any roots. Doing this, there will remain a certain number of equalities, and the number of the unknowns will be one less. Let us suppose that there still remains to determine several unknowns, each of which rises in the equations to several dimensions. In this case we can always combine two equalities together in such a way that it results in one where the unknown to determine will not have more than one dimension, and from there we will derive its value by a rational expression. Following this method, we will end up with two equalities that contain the last unknown quantity, and to whatever height it rises there, we have in algebra the means to develop them by way of combination with other equations, where the powers of the unknown will be successively decreased, and finally we will get one equation, in which will be found only the first power of the unknown, which will consequently be determined by a rational expression; which when substituted in the values of the other coefficients already found, will also furnish rational expressions for these. So when we will have found for u a real value, the values of all the other coefficients will become also necessarily real.

Theorem 6.

40. *Every equation of degree 16 is always resolvable into two real factors of degree 8.*

Proof

After eliminating the second term of the equation, it will have this form

$$x^{16} + Bx^{14} + Cx^{13} + Dx^{12} + \cdots = 0$$

and the number of the coefficients B, C, D, etc. will be 15. Assuming then its two factors of degree 8 to be

$$x^8 - ux^7 + \alpha x^6 + \beta x^5 + \gamma x^4 + \delta x^3 + \epsilon x^2 + \zeta x + \eta = 0,$$
$$x^8 + ux^7 + \theta x^6 + \iota x^5 + \chi x^4 + \lambda x^3 + \mu x^2 + \nu x + \xi = 0,$$

if we equate the product of these two factors to the given equation, we will obtain 15 equalities, from which it is necessary to look for the values of the coefficients u, α, β, γ, δ, etc., whose number is also 15, so that the problem is determinate. Therefore, if we from the start regard the coefficient u as known, we will have one equality more than the number of unknowns α, β, γ, etc., and so we will be able to derive from them their values determined by u and B, C, D, E, etc. without having to extract any roots. These values will therefore be rational and consequently also real, provided that we have a real value for the coefficient u. So it all comes down to showing that it is always possible to find a real value for the coefficient u. Now, having successively eliminated all the letters α, β, γ, δ, etc., we will finally reach one equation composed of the known coefficients B, C, D, E, etc. and of the unknown u, which will rise to a degree that we will derive by this reasoning. Since the quantity u indicates in general the sum of 8 arbitrary roots taken from 16 roots of the given equation, it is clear by the rules of combinations that the quantity u is susceptible to as many different values as there are units in this formula

$$\frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} = 12870.$$

Therefore, the equation which will determine the values of the unknown u will necessarily be of degree 12870. Since the sum of all 16 of the roots of the given equation is zero, if the sum of 8 arbitrary ones (that is to say one value of u) is p, then the sum of the 8 others will be $-p$, and so $-p$ is also a value of u. Indeed, if $u - p$ is a factor of the equation which determines u, then $u + p$ will also be a factor of it, and so their product $uu - pp$, containing the two roots p and $-p$ will also be a factor of it. Consequently, this equation is composed of $\frac{1}{2} \cdot 12870 = 6435$ factors of the form $uu - pp$, or it will be the product of factors such as

$$(uu - pp)(uu - qq)(uu - rr)(uu - ss) \cdots = 0,$$

the number of these factors being 6435. Since this number is odd, the last or constant term of this equation will be $-ppqqrrss$ etc. Putting then $pqrs$

etc. equal to P, it is certain that P is determinable by the coefficients B, C, D, E, etc., so that it is a rational function of them, and therefore real. So the last term of our equation, which must serve to determine u, will be equal to $-PP$, which is to say it will be strictly negative. From this it follows that this equation will necessarily have at least two real roots, one positive and the other negative, which consequently being taken for $+u$ and $-u$ will furnish two real factors of degree 8 for the given equation. Q.E.D.

Corollary 1.

41. So since each of these two factors of degree 8 is resolvable into four factors of the second degree, it is clear that every equation of degree 16 is resolvable into 8 double real factors; and an equation of degree 17, which certainly has a simple real factor, will have besides that 8 more double real factors.

Corollary 2.

42. The same resolvability into real factors, either simple or double, will also occur in all equations of a degree inferior to 16. For by multiplying such an equation by x or x^2 or x^3, etc. in order to elevate it to degree 16, we will look to it for its 8 double real factors, then by removing the factors x which were introduced by the multiplication, we will have the real factors of the given equation, which will be either simple or double.

Corollary 3.

43. It is therefore proved that every equation which does not surpass degree 17 is always resolvable into real factors, either simple or double.

Scholium

44. If we examine the force of these proofs, we will find that it consists in the final equation, which contains the sole unknown u, becoming of even degree, and in its last term being a negative square. This is what happened in the resolution of the equations of degree 4, 8, and 16. We will notice, furthermore, that the latter situation with the negative constant term cannot occur unless the degree exponent of the equation with u is an even number $2n$ such that half of it, n, is an odd number; because the last term is the product of n negative squares, and so it would become positive, if n were an even number. And this is the reason that our proof cannot be applied to equations of degree 12 or 20; because if we wanted to operate in this same manner on an equation for example of degree 20, by decomposing it into two factors of degree 10, as

$$x^{10} + ux^9 + \cdots \quad \text{and} \quad x^{10} - ux^9 + \cdots$$

after having eliminated the second term, we would see that the quantity u would have to be determined by an equation of degree

$$\frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} = 4 \cdot 11 \cdot 13 \cdot 17 \cdot 19,$$

half of which, being still an even number, would produce a positive final term for the equation, and we would no longer be able to draw the conclusion we need. Now, however much we reflect on this situation, we will find that the last term must necessarily become negative only when the given equation is of a degree where the exponent is a power of 2, and therefore the manner of proof that serves me here will have a place only for equations of degree 32, 64, 128, etc. But these cases are sufficient for our purpose, since having proved the resolvability into real factors for equations of whatever degree, it also follows for all the equations of a lesser degree.

Theorem 7.

45. *Every equation whose degree exponent is a binary power like 2^n (n being a whole number greater than 1) is resolvable into two real factors of degree 2^{n-1}.*

Proof

After eliminating the second term, the equation in question will be of this form

$$x^{2^n} + Bx^{2^n-2} + Cx^{2^n-3} + Dx^{2^n-4} + \cdots = 0,$$

where the number of coefficients B, C, D, etc. is $2^n - 1$. Let us now assume that the two factors sought are

$$x^{2^{n-1}} - ux^{2^{n-1}-1} + \alpha x^{2^{n-1}-2} + \beta x^{2^{n-1}-3} + \cdots = 0,$$
$$x^{2^{n-1}} + ux^{2^{n-1}-1} + \lambda x^{2^{n-1}-2} + \mu x^{2^{n-1}-3} + \cdots = 0,$$

where the number of coefficients to determine u, α, β, etc. is also $2^n - 1$. The comparison of the product of these two factors with the given equation furnishes just as many equalities, so that all the letters α, β, γ, etc. may be determined by a real expression of the knowns B, C, D, etc. and u, without extracting any roots. So in the end, to determine the unknown u, we will reach one equation which will have for its degree exponent

$$\frac{2^n(2^n-1)(2^n-2)(2^n-3)(2^n-4)\cdots(2^{n-1}+1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots 2^{n-1}},$$

as we know by the rules of combinations. Let N be this degree exponent of the equation for u, and by reversing the order of the factors in the denominator, we will have

$$N = \frac{2^n}{2^{n-1}} \cdot \frac{2^n - 1}{2^{n-1} - 1} \cdot \frac{2^n - 2}{2^{n-1} - 2} \cdot \frac{2^n - 3}{2^{n-1} - 3} \cdot \frac{2^n - 4}{2^{n-1} - 4} \cdots \frac{2^{n-1} + 1}{1},$$

and reducing each fraction to lowest terms:

$$N = 2 \cdot \frac{2^n - 1}{2^{n-1} - 1} \cdot \frac{2^{n-1} - 1}{2^{n-2} - 1} \cdot \frac{2^n - 3}{2^{n-1} - 3} \cdot \frac{2^{n-2} - 1}{2^{n-3} - 1} \cdots \frac{2^{n-1} + 1}{1}.$$

Now, it is certain that the number N is whole, and since both the product of the numerators and well as of the denominators are odd, this number will be oddly even, or its half an odd number. We will have, then, by starting with the last fraction

$$\tfrac{1}{2}N = \frac{2^{n-1} + 1}{1} \cdot \frac{2^{n-2} + 1}{1} \cdot \frac{2^{n-1} + 3}{3} \cdot \frac{2^{n-3} + 1}{1} \cdot \frac{2^{n-1} + 5}{5} \cdots \frac{2^n - 1}{2^{n-1} - 1}.$$

But since the second term of the given equation is absent, if p is one root u, then $-p$ will also be a root, and therefore $uu - pp$ a double factor, and the number of all the factors of this form will be equal to $\frac{1}{2}N$, which is to say an odd number. Consequently, the last term of the equation for u will be a negative square, which is an indicator that this equation contains at least two real values, one for u and the other for $-u$. From this we will form two real factors of degree 2^{n-1} of the given equation. Q.E.D.

Corollary 1.

46. Every equation, then, of degree 32 is resolvable into two real factors of degree 16, and therefore by the preceding theorem also resolvable into 16 real factors of the second degree. This must also be understood to hold for all the equations below degree 32, which we will be able to decompose into real factors, either simple or double.

Corollary 2.

47. Next, since every equation of degree 64 is resolvable into two real factors of degree 32, all equations which do not exceed degree 64 or 65 will also be resolvable into all real factors, each either simple or double.

Corollary 3.

48. In the same manner, we will extend this resolvability into real factors, either simple or double, successively to equations of degree 128, 256, 512, etc., so that it is now certain that every equation, however high its degree, is always resolvable into real factors, either simple or of the second degree.

Scholium

49. We see, then, a complete proof of the proposition that we commonly assume in analysis, and principally in integral calculus, by which we claim that every rational function of a variable x, as

$$x^m + Ax^{m-1} + Bx^{m-2} + \cdots$$

can always be resolved into real factors, either simple of the form $x + p$, or else double of the form $xx + px + q$. It is from the possibility of this resolution that we have derived this beautiful and important consequence: that the integral of a differential formula such as $\frac{Pdx}{Q}$, where P and Q indicate arbitrary rational functions of x, can always be expressed either algebraically, or by logarithms, or by arcs of the circle. Regarding the solidity of the proof I just gave of this beautiful property of equations, I believe that one will find no fault after carefully weighing the remarks which I have added. However, in case one wanted to have trouble recognizing the correctness of these proofs, I am going to add several propositions concerning this subject which will not depend on the preceding, and whose truth will serve to lift any doubt that one might still have.

Theorem 8.

50. *Every equation of degree 6 has at least one real factor of the second degree, independently of the preceding proofs.*

Proof

Let the given degree 6 equation be

$$x^6 + Ax^5 + Bx^4 + Cx^3 + Dx^4 + Ex + F = 0,$$

of which an arbitrary double factor is

$$xx - ux + v,$$

and the other factor will therefore be of the fourth degree, as

$$x^4 + \alpha x^3 + \beta x^2 + \gamma x + \delta,$$

and we understand that if one is real, the other must be real also. The product of these two factors must be equal to the given equation, so we will obtain 6 equalities from which it is necessary to determine the presumed coefficients u, v, α, β, γ, δ. This determination can be made, as I have already noted, by using rational expressions up until the last one, which will be of the coefficient u, from which it will be necessary to derive the value from an equation of a certain number of degrees. So if we are able to find a real value for u, those of the other coefficients α, β, γ, etc. would become also real, and therefore also the assumed factors themselves. It is a question, then, of considering the equation which will determine u, in order to see if it contains real values. Now, it is clear that in general u is the sum of two arbitrary roots of the given equation, and therefore it will be susceptible to as many different values as there are units in this formula: $\frac{6\cdot 5}{1\cdot 2} = 15$. So it is absolutely necessary that the equation for determining u contain 15 different values, neither more nor less, and thus this equation will be of degree 15, which is to say an odd degree. It will therefore surely have a real root, which when put for u will furnish us a real factor of the second degree, $xx - ux + v$, of the given degree 6 equation. Q.E.D.

Corollary

51. Every equation of degree 6 can always be resolved into two real factors, one of which is of the second degree, and the other of the fourth degree; and since the latter is resolvable into two real factors of the second degree, we will have three double real factors of the degree 6 equation.

Scholium

52. I assume here that it is possible to resolve an equation of the fourth degree into two double real factors, though my purpose is to render this proposition and the several that follow independent of the preceding proofs. Yet even when we would doubt their solidity, this doubt could center only upon the equations of degree 8, 16, etc., since the proof for the equations of the fourth degree is altogether accomplished, having even deduced the equation we need to determine the unknown u by algebraic operations, which could not be executed for the equations one degree higher, where it would be necessary to have recourse to some special principle. It is therefore remarkable that the resolution of a degree 6 equation is proved here by that of the fourth degree, as opposed to following the preceding theorems, when it was only permitted to recognize the possibility of this resolution after having proved it for equations of degree 8. So now we are convinced that every equation of degree 6 is resolvable into three double real factors, even when it would be impossible to similarly resolve the equations of degree 8. The method which serves me here for proving the resolution of equations of the sixth degree extends equally to all the equations whose degree exponent is oddly even, or whose half is an odd number, as I will show in the following theorem. As for the rest, we note here that also by virtue of this theorem every equation of the seventh degree is resolvable into a simple factor and three double factors, all real.

Theorem 9.

53. *Every equation whose degree exponent is a number of the form* $4n+2$ *always has at least one real factor of the second degree, and this independently of the above proofs.*

Proof

Since the given equation is of this form

$$x^{4n+2} + Ax^{4n+1} + Bx^{4n} + Cx^{4n-1} + \cdots = 0,$$

let one of its arbitrary double factors be

$$xx - ux + v,$$

and it is certain that u will be the sum of two arbitrary roots of the given equation. Since the number of all the roots is $4n+2$, if we combine two of them, the number of all the possible combinations will be

$$\frac{(4n+2)(4n+1)}{1 \cdot 2} = (2n+1)(4n+1)$$

and the letter u will be susceptible to that many values; or indeed u will be determined by an equation of degree $(2n+1)(4n+1)$, which being odd, this equation will necessarily have a real root which when put for u will give a double real factor $xx - ux + v$. From this it follows that every equation of degree $4n+2$ always has at least one real factor of the second degree. Q.E.D.

Corollary 1.

54. So if an equation of degree 8 is resolvable into four double real factors, every equation of degree 10 can be resolved into five double real factors, and to prove this we need no recourse to equations of degree 16, as we did previously.

Corollary 2.

55. And if every equation of degree 2^n is resolvable into 2^{n-1} double real factors, this theorem proves the resolvability into double real factors of equation of degree 2^n+2. And furthermore the equations of degrees 2^n+1 and 2^n+3 will also permit resolution into real factors, either simple or double, since being of odd degree, they have at least one simple real factor.

Theorem 10.

56. *Every equation whose degree exponent is a number of the form* $8n+4$ *has at least one real factor of the fourth degree, and this independently of the above proofs.*

Proof

If we set an arbitrary fourth-degree factor

$$x^4 - ux^3 + \alpha x^2 + \beta x + \gamma,$$

the coefficient u will be the sum of four arbitrary roots of the given equation. Since this equation has $8n+4$ roots, the number of all the possible values that the quantity u is susceptible to is

$$\frac{(8n+4)(8n+3)(8n+2)(8n+1)}{1\cdot 2\cdot 3\cdot 4} = \frac{(2n+1)(8n+3)(4n+1)(8n+1)}{3},$$

and therefore the quantity u will be determined by an equation of the same degree; and it is clear that the exponent of this degree, being a whole number, will be odd. So this equation will have at least one real root, which when put for u, will also determine real values for α, β, γ, and we will obtain a real factor of the fourth degree. Q.E.D.

Corollary 1.

57. Therefore, since a real factor of the fourth degree is incontestably resolvable into two real factors of the second degree, every equation of degree $8n+4$ will certainly have two double real factors at least, and the equations of degree $8n+5$ will have besides that one more simple real factor.

Corollary 2.

58. Since the equations of degree 12 are among this number, they will have two double real factors, and the third factor will be of degree 8. So if the latter is resolvable into four double real factors, we will have in total 6 double real factors, without needing to rise to equations of degree 16 in order to prove it.

Scholium

59. One will prove by similar reasoning that every equation of degree $16n+8$ has at least one real factor of degree 8, and one will pass moreover to equations of $32n+16$, $64n+32$, $128n+64$, etc. dimensions in order to prove that they have at least one real factor of degree 16, 32, 64, etc. From this we will derive this consequence: that all equations from degree 8 to degree 16 can be resolved into real factors, either simple or double, by assuming only the resolution of equations of degrees 4 and 8, and in general the resolution of every equation can be done without needing to reduce it to a higher degree, as we were obliged to do when using only the resolution of equations whose degrees were binary powers. So combining together these two ways of proving, we will no longer hesitate to agree with this general theorem, that every algebraic equation, of whatever degree, is always resolvable into real factors, either simple or double. However it is necessary to admit that it is for the most part impossible to execute this resolution, or to explicitly assign these real factors, because as soon as an equation passes the fourth degree, the rules of algebra are no longer sufficient to reveal to us these roots. But for the goal we have in view in establishing this general theorem, it suffices that we are assured that such a resolution is always possible, though we may never be able to execute it.

Theorem 11.

60. *If an algebraic equation, of whatever degree, has any imaginary roots, each one will be included in this general form $M + Ni$, the letters M and N indicating real quantities.*

Proof

Let the given arbitrary equation of degree n be

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \cdots = 0,$$

so that the number of all its roots is equal to n. We decompose this equation into all of its real factors, which will be either simple of the form $x - p = 0$, or of the second degree of the form $xx - 2px + q = 0$; and all the roots are found by solving the equalities that these factors furnish when set to zero. Each simple factor, or equation $x - p = 0$, gives a real root $x = p$. And each double factor, or equation $xx - 2px + q = 0$, contains two roots

$$x = p + \sqrt{pp - q} \quad \text{and} \quad x = p - \sqrt{pp - q},$$

which will also be real if $pp > q$. But if $pp < q$, let $q = pp + rr$, and we will have $\sqrt{pp - q} = \sqrt{-rr} = ri$. Therefore these two roots will be imaginary, namely

$$x = p + ri \quad \text{and} \quad x = p - ri.$$

So having demonstrated that it is always possible to resolve any equation into either simple or double real factors, all the roots will also be either real, or imaginary of the form $M + Ni$, where M and N are real quantities, so that the imaginary which enters into it is contained only in the form $\sqrt{-1}$. Q.E.D.

Corollary 1.

61. So if among the imaginary roots of an arbitrary equation is found $x = p + ri$, there will certainly also be found $x = p - ri$. This is clear both from the proof of this theorem, as well as from the nature of the radical sign $\sqrt{-1}$, which contains in an essential way the $+$ sign as well as the $-$ sign, so that knowing one imaginary root of an arbitrary equation, the other one reveals itself.

Corollary 2.

62. Having already shown that the number of all the imaginary roots that an arbitrary equation contains is even, each imaginary root $x = p + ri$ will have among the others its companion $x = p - ri$, which goes with it more than all the others, seeing that both the sum of these two roots, $2p$, as well as their product, $\sqrt{pp + rr}$, are real quantities.

Corollary 3.

63. From this it is also clear that if $x - p - ri$ is an imaginary factor of an arbitrary equation, the formula $x - p + ri$ will also be a factor of it. And these two factors joined together will give a double real factor of the same equation, which will be

$$xx - 2px + pp + rr.$$

Scholium

64. We understand from this, conversely, that if one would be able to show that all the imaginary roots of an arbitrary equation necessarily had the form $M + Ni$, it would be easy to show from this that every equation is also resolvable into real factors, either simple or of the second degree. For the real roots would always furnish as many simple real factors, and each imaginary root $x = p + ri$, when joined with its companion, $x = p - ri$, would produce a double real factor:

$$xx - 2px + pp + rr,$$

so that if an equation of degree $n = \alpha + 2\beta$ had α real roots and 2β imaginary roots, of which each were of the form $M + Ni$, it would be shown that this equation had α simple real factors and β double real factors. Now, it seems very plausible that every imaginary root, however complicated it might be, is always reducible to the form $M + Ni$, and Mr. d'Alembert has proved this in his excellent piece on integral calculus, which is found in the second volume of our *Mémoires*, in such a manner that there no longer remains the least doubt. However, as he used in his proof infinitely small quantities, though this consideration might not diminish the force of it, I will try to also derive from this source a rigorous proof of the general theorem to which this piece is intended, without having recourse to any infinitely small quantities. For this purpose I will have need of several preliminary theorems.

Theorem 12.

65. *Every function which is formed by addition, subtraction, multiplication, or division, from however many imaginary formulas of the form $M + Ni$, will always be included in the same form $M + Ni$, the letters M and N indicating real quantities.*

Proof

Let us imagine several imaginary formulas of the indicated form, these being

$$\alpha + \beta i \quad \text{and} \quad \gamma + \delta i \quad \text{and} \quad \epsilon + \zeta i \quad \text{and} \quad \eta + \theta i \quad \text{etc.}$$

and it is immediately clear that by adding these formulas together, or by taking some away, the resulting expression will always be included in the form $M + Ni$. It is also clear that if we multiply two or more of these formulas together, the product will always be contained in the form $M + Ni$. For the product of two,

$$\alpha + \beta i \quad \text{and} \quad \gamma + \delta i,$$

is

$$\alpha\gamma - \beta\delta + (\alpha\delta + \beta\gamma)i,$$

which when multiplied by $\epsilon + \zeta i$ will again give this form, and so on. So the only remaining question is division. But it is clear that this case always reduces to a fraction such as

$$\frac{A + Bi}{C + Di},$$

where both the numerator and the denominator are already made up of the first three operations of addition, subtraction, and multiplication, from as many imaginary formulas of the form $M + Ni$ as we wish. Now this fraction reduces to another, whose denominator is real, by multiplying above and below by $C - Di$. For then we will have

$$\frac{AC + BD + (BC - AD)i}{CC + DD},$$

so that putting M for $\frac{AC+BD}{CC+DD}$ and N for $\frac{BC-AD}{CC+DD}$, we will get the form $M + Ni$. Consequently, this form remains unaltered by whichever operations we use to join together as many imaginary formulas of the form $M + Ni$ as we wish.

Corollary 1.

66. From this it is also evident that all the powers whose exponent is a positive whole number, of an imaginary formula $A + Bi$, will always have the same form $M + Ni$, since these powers are formed by multiplication.

Corollary 2.

67. Next, since the power $(A + Bi)^n$ is included in the form $M + Ni$, if n is a positive whole number, the same form will apply if n is a negative whole number. For having

$$(A + Bi)^{-n} = \frac{1}{(A + Bi)^n} = \frac{1}{M + Ni},$$

this form reduces to

$$\frac{M - Ni}{MM + NN}.$$

Corollary 3.

68. The general form $M + Ni$ also includes all the real quantities, when we set $N = 0$. So by joining together by the four operations mentioned, not only imaginary formulas of the form $M + Ni$, but also reals, the product will always be included in the form $M + Ni$.

Corollary 4.

69. It can also happen that this product, although formed from imaginary formulas, becomes real, the imaginaries canceling each other, or rendering $N = 0$. Thus the product of $\alpha + \beta i$ by $\alpha - \beta i$ is real; and we know that $(-1 + \sqrt{-3})^3 = 8$.

Theorem 13.

70. *Whatever the power of the root we extract, either from a real quantity or from an imaginary of the form $M + Ni$, the roots will always be either real, or imaginary of the form $M + Ni$.*

Proof

Let n be the exponent of the root we need to extract, so that we have for consideration either of the values $\sqrt[n]{a}$ or $\sqrt[n]{a + bi}$. Now since the latter changes to the former if $b = 0$, it suffices to prove that

$$\sqrt[n]{a + bi} \quad \text{or} \quad (a + bi)^{1/n}$$

is included in the form $M + Ni$, however large the number n is. In order to prove this, we look for an angle φ such that its tangent is equal to b/a, or putting $\sqrt{aa + bb} = c$, we take the angle φ such that its sine is equal to b/c and the cosine is equal to a/c. We will then have

$$a + bi = c\,(\cos\varphi + i\sin\varphi),$$

since $\cos\varphi = a/c$ and $\sin\varphi = b/c$. Now, it was proved that an arbitrary power of such a form as

$$(\cos\varphi + i\sin\varphi)^m$$

is

$$\cos m\varphi + i\sin m\varphi$$

whatever number we mean by the letter m, whether it be positive, negative, whole, fractional, or even irrational. This set, we will have

$$\begin{aligned}(a + bi)^{1/n} = \sqrt[n]{a + bi} &= c^{1/n}(\cos\varphi + i\sin\varphi)^{1/n}\\ &= \left(\cos\frac{1}{n}\varphi + i\sin\frac{1}{n}\varphi\right)\sqrt[n]{c}.\end{aligned}$$

So since $c = \sqrt{aa + bb}$ is a real quantity and positive, the angle φ and therefore also its part $\frac{1}{n}\varphi$ with its sine and cosine are also real quantities, and it is evident that

$$\sqrt[n]{a + bi}$$

or

$$\left(\cos\frac{1}{n}\varphi + i\sin\frac{1}{n}\varphi\right)\sqrt[n]{c}$$

belongs to the form $M + Ni$. So all the roots of a real quantity, or an imaginary quantity of the form $M + Ni$, are always included in the general form $M + Ni$. Q.E.D

Corollary 1.

71. As we know that every quantity has two square roots, three cube roots, four fourth roots, and so on, we find by this method all the roots, of which the number is n, since $\frac{1}{n}\varphi$ has as many different values.

Corollary 2.

72. For since φ is the angle whose sine is b/c and whose cosine is a/c, in place of φ we can also take the angles

$$4\rho + \varphi, \quad 8\rho + \varphi, \quad 12\rho + \varphi, \quad \text{etc.}$$

where ρ indicates the right angle, since all these angles have the same sine and cosine. So substituting for φ in the root found

$$\left(\cos\frac{1}{n}\varphi + i\sin\frac{1}{n}\varphi\right)\sqrt[n]{c}$$

these angles φ, $4\rho + \varphi$, $8\rho + \varphi$, $12\rho + \varphi$, etc., we will find as many different expressions as there are units in the exponent n.

Corollary 3.

73. Since n can indicate an arbitrary number, it follows from our proof that not only $\sqrt[n]{a + bi}$, where n is a positive whole number, but in general that this expression $(a + bi)^m$, whatever the number indicated by m, either positive, or negative, or whole, or fractional, or even irrational, is always included in the general form $M + Ni$.

Corollary 4.

74. Consequently, not only the four arithmetic operations, but also the extraction of roots of whatever degree, does not change the form $M + Ni$ of the imaginary quantities, when we apply them in an arbitrary manner.

Scholium

75. If the quantity for which we seek all the roots of a certain degree is real, or $b = 0$, we will have $c = \sqrt{aa}$, from which we will get a positive value for c, even when a is negative; and the angle φ will be equal to either 0 or 180°, according to whether the cosine, a/c, is $+1$ or -1. In the first case, where a is positive and $c = a$, the values of φ will therefore be 0, 4ρ, 8ρ, 12ρ, etc. and the roots of degree n of the number a will be, putting ρ for the mark of a right angle,

$$\sqrt[n]{a}, \quad \left(\cos\frac{4\rho}{n} + i\sin\frac{4\rho}{n}\right)\sqrt[n]{a}, \quad \left(\cos\frac{8\rho}{n} + i\sin\frac{8\rho}{n}\right)\sqrt[n]{a}, \quad \text{etc.}$$

Now, if a is a negative number, we will have the following expressions, or else the values of $\sqrt[n]{-a}$ will be

$$\left(\cos\frac{2\rho}{n} + i\sin\frac{2\rho}{n}\right)\sqrt[n]{a}, \quad \left(\cos\frac{6\rho}{n} + i\sin\frac{6\rho}{n}\right)\sqrt[n]{a}, \quad \text{etc.},$$

substituting for φ successively 2ρ, 6ρ, 10ρ, 14ρ, etc. But since this matter has already been sufficiently developed, I will limit myself here to this unique consequence: that the extraction of roots, both of real quantities and imaginaries of the form $M + Ni$, always produces either real quantities, or imaginary quantities of the form $M + Ni$.

Theorem 14.

76. *Whatever the degree of an algebraic equation, all the imaginary roots that it can have are always included in this general form $M + Ni$, so that M and N are real quantities.*

Proof

Let the given general equation be

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \cdots = 0,$$

and although we are not in a position to assign a general formula which contains all the roots, like we can for equations of the second, third, and fourth degrees, it is nevertheless certain that this formula would be composed of several radical signs, in which the known quantities A, B, C, D, E, etc. will be entangled. We can also note that this analytic expression of an arbitrary root will contain several members, each of which will be the root of a certain degree of a quantity, which yet again contains radical signs, and that the latter will have after them others still, and so on, until we reach, for each member, the last radical sign, which will not modify anything but real quantities. Let us successively climb out of these last radical signs, and it is clear that the quantity marked by the last radical sign will be either real or imaginary of the form $M + Ni$. Then in front of this quantity, joined with some value, either real or imaginary also of the form $M + Ni$, will be found a new radical sign, which therefore reduces to $\sqrt[n]{M + Ni}$, whose value is yet again of the form $M + Ni$; and if we climb in this manner up to the first radical signs, which distinguish the members, we will see that no operation can separate us from this form, and that consequently each member will finally have the same form, no matter how large the number of radical signs which envelop them. From this it follows that the general expression, which contains all the roots of the given equation, will necessarily reduce to the form $M + Ni$, so that all the imaginary roots would have only that form. Q.E.D.

Scholium 1.

77. We see, then, a new proof of the general theorem that I proposed to prove here, to which one could find nothing to object, unless it is that we do not know how the roots of equations of powers higher than the fourth are entangled. But this objection will not have any force, provided that we agree that the expressions for the roots contain nothing other than the operations of extraction of roots, in addition to the four common operations, and one could not hold that any transcendental operations are involved. But if the conjecture that I advanced in the past, on the form of roots of equations of an arbitrary order, is well-founded, the proof that I just gave here would have all the force that one could wish. For given an arbitrary equation of degree n, I say that there will always be an equation of degree $n - 1$, whose $n - 1$ roots are α, β, γ, δ, ϵ, etc., such that an arbitrary root of the other equation of degree n will be

$$a + \sqrt[n]{\alpha} + \sqrt[n]{\beta} + \sqrt[n]{\gamma} + \sqrt[n]{\delta} + \cdots,$$

where a is a real quantity. So, if the roots of the equation of degree $n-1$ are either real or of the form $M + Ni$, the roots of the equation of degree n will also have this form. Consequently, since the roots of equations of the second degree are either real or of the form $M + Ni$, the roots of equations of the third degree also reduce to this form, and therefore also roots of equations of degrees 4, 5, 6, etc., to infinity.

Scholium 2.

78. From this we will again draw this important consequence: that any imaginary quantity, however complicated it may be, is always reducible to this formula $M + Ni$, so that every imaginary quantity is always composed of two numbers, one of which is a real quantity indicated by M, and the other the product of another real quantity N, multiplied by $\sqrt{-1}$, so that $\sqrt{-1}$ is the sole source of all the imaginary expressions. For if we look to the origin of imaginary quantities, which is the extraction of roots or the resolution of equations, it has been shown that all the imaginary quantities which result from them are always included in this form $M + Ni$, and furthermore I have shown that whatever the manner we treat one or several imaginary quantities of this form using the operations of analysis—addition, subtraction, multiplication, division, and the extraction of roots—all the resulting expressions always reduce to the same form $M + Ni$. We will be obliged, then, to agree with this truth, insofar as only algebraic operations are involved in the imaginary formulas. But one will doubt, perhaps, whether the imaginary quantities which originate from transcendental operations, such as those which surround the nature of logarithms or of angles, are still reducible to the same form. So in order to lift this doubt, I will show that all the transcendental operations which are known do not produce imaginary quantities that diverge from the indicated form. And although it would be impossible to apply the same reasoning to all the transcendental operations, the following propositions will remove all remaining cause to doubt the truth of this general property of all the imaginary quantities, from whatever source they may draw their origin.

Problem 1.

79. *An imaginary quantity being raised to a power where the exponent is an arbitrary real quantity, to determine the imaginary form which results.*

Solution

Let $a+bi$ be the imaginary quantity, and m the real exponent of the power, so that it is a question of determining M and N in order that

$$(a + bi)^m = M + Ni.$$

Let us set $\sqrt{aa+bb}=c$, and c will always be a real quantity and positive, for we do not consider here the ambiguity of the sign $\sqrt{\ }$. Next, let us seek the angle φ such that its sine is b/c and the cosine is a/c, taking into account here the nature of the quantities a and b, whether they are positive or negative. It is certain that we will always be able to assign this angle φ, whatever the quantities a and b, provided they are real, which we assume. Now having found this angle φ, which will always be real, we will have at the same time all the other angles whose sine, b/c, and cosine, a/c, are the same; for putting π for the angle of $180°$, all these angles will be

$$\varphi, \quad 2\pi+\varphi, \quad 4\pi+\varphi, \quad 6\pi+\varphi, \quad 8\pi+\varphi, \quad \text{etc.},$$

to which we can add these

$$-2\pi+\varphi, \quad -4\pi+\varphi, \quad -6\pi+\varphi, \quad -8\pi+\varphi, \quad \text{etc.}$$

This set, we will have

$$a+bi=c\,(\cos\varphi+i\sin\varphi),$$

and the given power

$$(a+bi)^m=c^m(\cos\varphi+i\sin\varphi)^m,$$

where c^m will always have one real positive value, which it is necessary to give it in preference to all the other values it could have. Then it is proved that

$$(\cos\varphi+i\sin\varphi)^m=\cos m\varphi+i\sin m\varphi,$$

where it must be noted that since m is a real quantity, the angle $m\varphi$ will also be real, and therefore also its sine and cosine. Therefore we will have

$$(a+bi)^m=c^m(\cos m\varphi+i\sin m\varphi),$$

or indeed the power $(a+bi)^m$ is contained in the form $M+Ni$, by taking

$$M=c^m\cos m\varphi \quad \text{and} \quad N=c^m\sin m\varphi,$$

where

$$c=\sqrt{aa+bb} \quad \text{and} \quad \cos\varphi=a/c \quad \text{and} \quad \sin\varphi=b/c.$$

Q.E.D.

Corollary 1.

80. In the same manner that

$$(\cos\varphi + i\sin\varphi)^m = \cos m\varphi + i\sin m\varphi,$$

it will also be that

$$(\cos\varphi - i\sin\varphi)^m = \cos m\varphi - i\sin m\varphi,$$

and therefore we will have

$$(a - bi)^m = c^m(\cos m\varphi - i\sin m\varphi),$$

where φ indicates the same angle as in the preceding.

Corollary 2.

81. If the exponent m is negative, since

$$\sin(-m\varphi) = -\sin m\varphi \quad \text{and} \quad \cos(-m\varphi) = \cos m\varphi,$$

it will also be that

$$(\cos\varphi \pm i\sin\varphi)^{-m} = \cos m\varphi \mp i\sin m\varphi$$

and

$$(a \pm bi)^{-m} = c^{-m}(\cos m\varphi \mp i\sin m\varphi).$$

Corollary 3.

82. If m is a whole number, either negative or positive, the formula $(a+bi)^m$ will have only a single value; for although we substitute for φ all the angles φ, $\pm 2\pi + \varphi$, $\pm 4\pi + \varphi$, $\pm 6\pi + \varphi$, etc., we will always find for $\sin m\varphi$ and $\cos m\varphi$ the same values.

Corollary 4.

83. But if the exponent m is a fractional number μ/ν, then the expression $(a + bi)^{\mu/\nu}$ will have as many different values as there are units in ν; for by substituting for φ the indicated angles, we will obtain as many different values for $\sin m\varphi$ and $\cos m\varphi$, as the number ν contains units.

Corollary 5.

84. From this it is clear that if m is an irrational number, incommensurable with unity, the expression $(a + bi)^m$ will also have an infinity of different values, because all the angles φ, $\pm 2\pi + \varphi$, $\pm 4\pi + \varphi$, $\pm 6\pi + \varphi$, etc. will supply distinct values for $\sin m\varphi$ and $\cos m\varphi$.

Scholium 1.

85. The basis of the solution to this problem is

$$(\cos\varphi + i\sin\varphi)^m = \cos m\varphi + i\sin m\varphi,$$

whose truth is proved by the known theorems on the multiplication of angles. For having two arbitrary angles φ and θ, it will be that

$$(\cos\varphi + i\sin\varphi)(\cos\theta + i\sin\theta) = \cos(\varphi+\theta) + i\sin(\varphi+\theta),$$

which is clear by explicit multiplication, which gives

$$\cos\varphi\cos\theta - \sin\varphi\sin\theta + (\cos\varphi\sin\theta + \sin\varphi\cos\theta)\, i.$$

Now, we know that

$$\cos\varphi\cos\theta - \sin\varphi\sin\theta = \cos(\varphi+\theta)$$

and

$$\cos\varphi\sin\theta + \sin\varphi\cos\theta = \sin(\varphi+\theta).$$

From this we easily derive the consequence, which is

$$(\cos\varphi + i\sin\varphi)^m = \cos m\varphi + i\sin m\varphi,$$

when the exponent m is a whole number. But that the same formula also has a place when m is an arbitrary number, differentiating after taking the logarithms will put away any doubt. For, taking logarithms, we will have

$$m\ln(\cos\varphi + i\sin\varphi) = \ln(\cos m\varphi + i\sin m\varphi).$$

Now, treating the angle φ as a variable quantity, we will have

$$\frac{-m\,d\varphi\,\sin\varphi + m\,d\varphi\,i\cos\varphi}{\cos\varphi + i\sin\varphi} = \frac{-m\,d\varphi\,\sin m\varphi + m\,d\varphi\,i\cos m\varphi}{\cos m\varphi + i\sin m\varphi},$$

and multiplying the numerators by $-i$, we will obtain

$$\frac{-m\,d\varphi\,(\cos\varphi + i\sin\varphi)}{\cos\varphi + i\sin\varphi} = \frac{m\,d\varphi\,(\cos m\varphi + i\sin m\varphi)}{\cos m\varphi + i\sin m\varphi} = m\,d\varphi,$$

which is an identity equation.

Scholium 2.

86. But one will ask how we would have been able to arrive at the transformation of the formula $(a+bi)^m$ to the form $M+Ni$, if we did not know the given property of multiple angles, which seems at first altogether foreign to this goal. For this purpose I will adjoin here another solution of the problem, without using this property; and the method which will serve me will also lead to the solution of the problems which follow. As it is a question, then, of converting the form $(a+bi)^m$ into that of $(x+yi)$, I put

$$(a+bi)^m = x + yi$$

and, taking the logarithms, we will have

$$m\ln(a+bi) = \ln(x+yi).$$

Now regarding a, b, x, and y as variables, I take the differentials

$$\frac{m\,da + m\,db\,i}{a+bi} = \frac{dx\ + dy\,i}{x+yi},$$

which reduce to this equation

$$\frac{ma\,da - mb\,da\,i + ma\,db\,i + mb\,db}{aa+bb} = \frac{x\,dx + x\,dy\,i - y\,dx\,i + y\,dy}{xx+yy},$$

where it is necessary that the real and imaginary members be separately set equal to each other, since it is impossible to equate a real quantity to an imaginary. From this we will derive two equations:

$$\frac{ma\,da + mb\,db}{aa+bb} = \frac{x\,dx + y\,dy}{xx+yy}$$

and

$$\frac{m(a\,db - b\,da)}{aa+bb} = \frac{x\,dy - y\,dx}{xx+yy}.$$

The integral of the first is

$$m \ln \sqrt{aa + bb} = \ln \sqrt{xx + yy} + \ln C.$$

So let $\sqrt{aa + bb} = c$, and we will have

$$c^m = C\sqrt{xx + yy}.$$

In order to determine this constant C, we only have to note that if $b = 0$ and $a = 1$, it is necessary that $y = 0$ and $x = 1$, from which we see that it must be that $C = 1$. So setting $\sqrt{aa + bb} = c$, we will have

$$\sqrt{xx + yy} = c^m.$$

Next, the integral of the other equation is

$$m \arctan \frac{b}{a} = \arctan \frac{y}{x} + C,$$

where we see that the constant C must be equal to 0, since if $b = 0$, it must also be that $y = 0$. Consequently, we will have

$$m \arctan \frac{b}{a} = \arctan \frac{y}{x}.$$

Let φ be the angle whose tangent is b/a, or as well $\sin \varphi = b/c$ and $\cos \varphi = a/c$, and having $m\varphi = \arctan \frac{y}{x}$, it will be that

$$\frac{y}{x} = \tan m\varphi,$$

or

$$\frac{y}{\sqrt{xx + yy}} = \sin m\varphi \quad \text{and} \quad \frac{x}{\sqrt{xx + yy}} = \cos m\varphi.$$

So since $\sqrt{xx + yy} = c^m$, we will have for the solution of the problem

$$x = c^m \cos m\varphi \quad \text{and} \quad y = c^m \sin m\varphi,$$

taking $c = \sqrt{aa + bb}$, and the angle φ being such that $\sin \varphi = b/c$ and $\cos \varphi = a/c$. From this we see that the angle φ has an infinity of values, as I have already noted, which are φ, $\pm 2\pi + \varphi$, $\pm 4\pi + \varphi$, $\pm 6\pi + \varphi$, etc.

Problem 2.

87. *A positive real quantity being raised to a power where the exponent is an imaginary quantity, to find the imaginary value of this power.*

Solution

Let a be a positive real quantity and $m+ni$ the exponent of the power, so that it is necessary to seek the imaginary value of a^{m+ni}. So let

$$a^{m+ni} = x + yi,$$

and we will have

$$(m+ni)\ln a = \ln(x+yi),$$

taking the differentials of which while putting a, x, and y as variables, we will have

$$\frac{m\,da}{a} + \frac{n\,da\,i}{a} = \frac{dx\ + dy\,i}{x+yi} = \frac{x\,dx + y\,dy}{xx+yy} + \frac{x\,dy - y\,dx}{xx+yy}\,i.$$

Then separately equating the real and imaginary members, we will have these two equations

$$\frac{m\,da}{a} = \frac{x\,dx + y\,dy}{xx+yy} \quad \text{and} \quad \frac{n\,da}{a} = \frac{x\,dy - y\,dx}{xx+yy},$$

of which the integrals taken, as is necessary, will be

$$\sqrt{xx+yy} = a^m$$

and

$$\arctan\frac{y}{x} = n\ln a \quad \text{or} \quad \frac{y}{x} = \tan(n\ln a)$$

where $\ln a$ indicates the hyperbolic logarithm of the positive real quantity a, which will consequently also have a real value. Then by taking, in a circle of radius 1, an arc equal to $n\ln a$, because $\sqrt{xx+yy} = a^m$, we will obtain

$$x = a^m\cos(n\ln a) \quad \text{and} \quad y = a^m\sin(n\ln a),$$

and these values being put for x and y, we will have

$$a^{m+ni} = x + yi.$$

Q.E.D.

Corollary 1.

88. The imaginary quantity a^{m+ni} will therefore also be included in the general form $M + Ni$, since we just found

$$a^{m+ni} = a^m \cos(n \ln a) + a^m \, i \sin(n \ln a),$$

when a is a positive real quantity; for if a were a negative quantity, though real, its logarithm would be imaginary, and therefore both $\cos(n \ln a)$ and $\sin(n \ln a)$ would also be imaginary.

Corollary 2.

89. Since $a^{m+ni} = a^m \cdot a^{ni}$, we will have

$$a^{ni} = \cos(n \ln a) + i \sin(n \ln a),$$

and taking n negative, we will also have

$$a^{-ni} = \cos(n \ln a) - i \sin(n \ln a).$$

Corollary 3.

90. From this it follows that the following formulas are real:

$$\frac{a^{ni} + a^{-ni}}{2} = \cos(n \ln a)$$

and

$$\frac{a^{ni} - a^{-ni}}{2i} = \sin(n \ln a).$$

Now, if $a = 1$ we will have both

$$1^{ni} = 1 \quad \text{and} \quad 1^{-ni} = 1$$

because $\ln 1 = 0$.

Corollary 4.

91. So if we put $n = 1$ and $a = 2$, we will have

$$\frac{2^i + 2^{-i}}{2} = \cos(\ln 2)$$

and

$$\frac{2^i - 2^{-i}}{2i} = \sin(\ln 2).$$

Now since

$$\ln 2 = 0.6931471805599,$$

we will derive from it

$$\cos(\ln 2) = 0.7692389013540 = \frac{2^i + 2^{-i}}{2}.$$

[Mais l'arc même dont le cosinus = ln 2 se trouve 39° 42^{I} 51^{II} 52^{III} 9^{IV}.]

Scholium 2.

92. The case where a is a negative number, which is not included in this solution, although a is a real quantity, is solved by the next problem, where I will take for the quantity which must be raised to an imaginary exponent, an arbitrary imaginary quantity of the form $a + bi$, which encompasses, by setting $b = 0$, all real quantities, both negative and positive.

Problem 3.

93. *An imaginary quantity being raised to a power where the exponent is also imaginary, to find the imaginary value of this power.*

Solution

Let $a+bi$ be the imaginary quantity, and $m+ni$ the exponent of the power, so that it is necessary to find the value of this formula:

$$(a + bi)^{m+ni}$$

Let us therefore put for this purpose

$$(a + bi)^{m+ni} = x + yi,$$

and, taking logarithms, we will have

$$(m + ni) \ln(a + bi) = \ln(x + yi).$$

Let us pass to the differentials, and since, as we have already seen,

$$d. \ln(x + yi) = \frac{x\,dx + y\,dy}{xx + yy} + \frac{x\,dy - y\,dx}{xx + yy}\, i,$$

we will have

$$\frac{m(a\,da+b\,db)}{aa+bb}+\frac{n(a\,da+b\,db)}{aa+bb}\,i+\frac{m(a\,db-b\,da)}{aa+bb}\,i-\frac{n(a\,db-b\,da)}{aa+bb}$$
$$=\frac{x\,dx+y\,dy}{xx+yy}+\frac{(x\,dy-y\,dx)}{xx+yy}\,i.$$

We now separately equate the real and imaginary members, obtaining these two equalities:

$$\frac{m(a\,da+b\,db)}{aa+bb}-\frac{n(a\,db-b\,da)}{aa+bb}=\frac{x\,dx+y\,dy}{xx+yy},$$
$$\frac{m(a\,db-b\,da)}{aa+bb}+\frac{n(a\,da+b\,db)}{aa+bb}=\frac{x\,dy-y\,dx}{xx+yy}.$$

In order to take the integrals let

$$\sqrt{aa+bb}=c \quad\text{and}\quad \arctan\frac{b}{a}=\varphi,$$

or indeed

$$\sin\varphi=\frac{b}{c} \quad\text{and}\quad \cos\varphi=\frac{a}{c},$$

from which we can always find the angle φ. Now, I assume here that c is a positive quantity $\sqrt{aa+bb}$. This noted, our integrals will be

$$m\ln c-n\varphi=\ln\sqrt{xx+yy},$$
$$m\varphi+n\ln c=\arctan\frac{y}{x}.$$

From this it follows that we will have

$$\sqrt{xx+yy}=c^m e^{-n\varphi}$$

putting e for the number whose hyperbolic logarithm is equal to 1. Thus, in order to find the values x and y of this equation:

$$(a+bi)^{m+ni}=x+yi,$$

having set $c = \sqrt{aa + bb}$ and taken the angle φ such that $\cos\varphi = a/c$ and $\sin\varphi = b/c$, we will have

$$x = c^m e^{-n\varphi} \cos(m\varphi + n \ln c),$$
$$y = c^m e^{-n\varphi} \sin(m\varphi + n \ln c).$$

Q.E.D.

Corollary 1.

94. If $b = 0$ and a is a positive quantity, we will have $c = a$ and the angle $\varphi = 0$ or $\pm 2\pi$ or $\pm 4\pi$, or in general $\varphi = 2\lambda\pi$, taking λ for an arbitrary whole number. So we will have

$$a^{m+ni} = a^m e^{-2\lambda n\pi}\big(\cos(2\lambda m\pi + n \ln a) + i \sin(2\lambda m\pi + n \ln a)\big),$$

which agrees with the preceding form when $\lambda = 0$, so that this transformation is more general.

Corollary 2.

95. If $b = 0$ and a is a negative quantity $-a$, it will still be $c = +a$ and, because $\cos\varphi = -a/c = -1$, the angle φ will be $\pm\pi$ or $\pm 3\pi$ or $\pm 5\pi$ etc., or in general $\varphi = (2\lambda - 1)\pi$, taking for λ an arbitrary whole number, either positive or negative. We will then have

$$(-a)^{m+ni} = a^m e^{-(2\lambda-1)n\pi}\big(\cos((2\lambda - 1)m\pi + n \ln a) + i \sin((2\lambda - 1)m\pi + n \ln a)\big).$$

Corollary 3.

96. In general then, whatever the quantities a and b are, by giving to c the positive value of $\sqrt{aa + bb}$, and taking for φ an angle such that $\sin\varphi = b/c$ and $\cos\varphi = a/c$, since for φ we can equally take in general the angle $2\lambda + \varphi$, where λ indicates an arbitrary positive or negative whole number, we will have

$$(a + bi)^{m+ni} = c^m e^{-2\lambda n\pi - n\varphi}\big(\cos(2\lambda m\pi + m\varphi + n \ln c) + i \sin(2\lambda m\pi + m\varphi + n \ln c)\big),$$

from which we will find all the possible values that this formula

$$(a + bi)^{m+ni}$$

contains, by successively giving to λ all the values $0, \pm 1, \pm 2, \pm 3, \pm 4$, etc., where it suffices to take for c^m the sole real positive value that is included in it.

Corollary 4.

97. If $a = 0$, $m = 0$, and $b = 1$, we will have $c = 1$ and $\varphi = \frac{1}{2}\pi$, from which we will derive this transformation

$$i^{ni} = e^{-2\lambda n\pi - \frac{1}{2}n\pi}$$

or indeed

$$i^i = e^{-2\lambda\pi - \frac{1}{2}\pi}$$

which is all the more remarkable because it is real, and it even contains an infinity of different real values. For setting $\lambda = 0$, we will have in numbers

$$i^i = 0.2078795763507.$$

Corollary 5.

98. If we set $a = \cos\varphi$ and $b = \sin\varphi$, taking $c = 1$, so that $\ln c = 0$, we will have this remarkable transformation

$$(\cos\varphi + i\sin\varphi)^{m+ni} = e^{-2\lambda n\pi - n\varphi}\big(\cos m(2\lambda\pi + \varphi) + i\sin m(2\lambda\pi + \varphi)\big),$$

and if $m = 0$, this formula will be real:

$$(\cos\varphi + i\sin\varphi)^{ni} = e^{-2\lambda n\pi - n\varphi}$$

Scholium

99. We see from this, then, that all imaginary quantities which draw their origin not only from algebraic operations, but also those which arise from the raising to arbitrary, and even imaginary, exponents are always reducible to the general form

$$M + Ni.$$

And we also understand from this that if the exponents were themselves of such powers to imaginary exponents, the value of the whole formula would nevertheless be included in the form $M + Ni$. For it is clear that if α, β, γ indicate imaginary quantities of the form $M + Ni$, then the derived quantity

α^{β^γ} would also always be included in that form, since the exponent β^γ is reducible to this form.

Problem 4.

100. *Given an arbitrary imaginary number, to find its hyperbolic logarithm.*

Solution

Let $a+bi$ be the imaginary quantity whose logarithm we must find, which will be $x+yi$, so that

$$\ln(a+bi) = x+yi.$$

Taking the differentials, we will have

$$\frac{a\,da+b\,db}{aa+bb}+\frac{a\,db-b\,da}{aa+bb}\,i = dx+i\,dy.$$

Again let $\sqrt{a+bb}=c$, and the angle φ such that $\cos\varphi = a/c$ and $\sin\varphi = b/c$, and by integration we will find

$$x=\ln\sqrt{aa+bb}=\ln c \quad \text{and} \quad y=\arctan\frac{b}{a}=\varphi.$$

So we will have

$$\ln(a+bi)=\ln\sqrt{aa+bb}+i\cdot\arccos\frac{a}{\sqrt{aa+bb}}$$

or

$$\ln(a+bi)=\ln\sqrt{aa+bb}+i\cdot\arcsin\frac{b}{\sqrt{aa+bb}}.$$

Q.E.D.

Corollary 1.

101. Since there are infinitely many angles which correspond to the same sine, $b/\sqrt{aa+bb}$, and cosine, $a/\sqrt{aa+bb}$, each number, real as well as imaginary, has an infinity of logarithms, all of which are imaginary except for a single one, when $b=0$ and a is a positive number.

Corollary 2.

102. If we put $\sqrt{aa+bb}=c$, and the angle found is φ, because $a=c\cos\varphi$ and $b=c\sin\varphi$, we will have

$$\ln\big(c\,(\cos\varphi+i\sin\varphi)\big)=\ln c+i\varphi,$$

where in place of φ it is permitted to put $\pm 2\pi + \varphi$, $\pm 4\pi + \varphi$, $\pm 6\pi + \varphi$, etc., the character π indicating the sum of two right angles. We will therefore have

$$\ln(\cos\varphi + i\sin\varphi) = i\varphi.$$

Problem 5.

103. *Given an imaginary logarithm, to find the number which goes with it.*

Solution

Let $a + bi$ be the given imaginary logarithm, and $x + yi$ the number which goes with it, so that

$$\ln(x + yi) = a + bi.$$

Comparing this equation to what we just deduced in the preceding article:

$$\ln\big(c\,(\cos\varphi + i\sin\varphi)\big) = \ln c + i\varphi,$$

we will have $\varphi = b$ and $\ln c = a$, so $c = e^a$ assuming $\ln e = 1$. From this we will derive

$$x = e^a \cos b \quad \text{and} \quad y = e^a \sin b.$$

Consequently, the number which corresponds to the logarithm $a + bi$ will be equal to

$$e^a(\cos b + i\sin b).$$

Q.E.D.

Corollary 1.

104. So every time that b is either zero, or equal to $\pm\pi$ or $\pm 2\pi$ or $\pm 3\pi$ or in general $b = \pm\lambda\pi$, the number which corresponds to the logarithm $a + bi$ will be real and equal to $\pm e^a$. It will be positive if λ is an even number, and negative if λ is odd.

Corollary 2.

105. We also see that there is but a single number which leads to a given logarithm, and whenever the logarithm is real, the number will also be real. But there are also cases where, although the logarithm is imaginary, the number is nevertheless real. But since I have already sufficiently explained this matter elsewhere, I will proceed to imaginary quantities, which include angles, or their sines, cosines, and tangents.

Problem 6.

106. *Given an angle, or an arbitrary imaginary arc of the circle, to find its sine and cosine and tangent.*

Solution

Let $a+bi$ be the given angle, which being made up of two parts, one real, a, and the other imaginary, bi, we only have to look for the sine and the cosine of this imaginary arc. Now, the known series gives us:

$$\cos bi = 1 + \frac{bb}{1\cdot 2} + \frac{b^4}{1\cdot 2\cdot 3\cdot 4} + \cdots = \frac{e^b+e^{-b}}{2},$$
$$\sin bi = bi + \frac{b^3 i}{1\cdot 2\cdot 3} + \frac{b^5 i}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots = \frac{e^b-e^{-b}}{2}\, i,$$

and from this we will derive

$$\sin(a+bi) = \frac{1}{2}(e^b+e^{-b})\sin a + \frac{i}{2}(e^b-e^{-b})\cos a,$$
$$\cos(a+bi) = \frac{1}{2}(e^b+e^{-b})\cos a - \frac{i}{2}(e^b-e^{-b})\sin a,$$

so the tangent will be

$$\tan(a+bi) = \frac{(e^b+e^{-b})\tan a + i\,(e^b-e^{-b})}{(e^b+e^{-b}) - i\,(e^b-e^{-b})\tan a}$$

or

$$\tan(a+bi) = \frac{(e^{2b}+1)\tan a + i\,(e^{2b}-1)}{(e^{2b}+1) - i\,(e^{2b}-1)\tan a}\,.$$

Q.E.D.

Corollary 1.

107. Since in the expression of the tangent, both the numerator and the denominator are imaginaries, if we free the denominator of them, we will have

$$\tan(a+bi) = \frac{2e^{2b}\sin 2a + i\,(e^{4b}-1)}{e^{4b}+2e^{2b}\cos 2a + 1}\,.$$

Corollary 2.

108. The sine of the angle $a+bi$ becomes real not only in the case $b=0$, where the angle itself is real, but also in the case where $\cos a = 0$, which happens when, putting ρ to indicate a right angle, we have $a = (2\lambda-1)\rho$, where λ signifies a whole number, either positive or negative. For then we will have

$$\sin((2\lambda-1)\rho+bi)=\pm\tfrac{1}{2}(e^b+e^{-b}),$$

where the $+$ sign occurs if λ is an odd number, and the $-$ sign if λ is an even number.

Corollary 3.

109. In the same way, the cosine of the angle $a+bi$ will be real not only when $b=0$, but also when $\sin a=0$, which happens if $a=2\lambda\rho$, and then we will have

$$\cos(2\lambda\rho+bi)=\pm\tfrac{1}{2}(e^b+e^{-b}),$$

where the top $+$ sign happens if λ is an even number, and the bottom $-$ sign if λ is an odd number.

Corollary 4.

110. As for the tangent of the angle $a+bi$, it can never become real unless $b=0$, in which case the angle itself is real.

Corollary 5.

111. The formulas we found will yet again furnish, by giving to $\sqrt{-1}$ its two signs, which equally suit it, the following formulas, which it will be appropriate to note:

$$\begin{aligned}
\sin(a+bi)+\sin(a-bi)&=(e^b+e^{-b})\sin a,\\
\sin(a+bi)-\sin(a-bi)&=(e^b-e^{-b})\,i\cos a,\\
\cos(a+bi)+\cos(a-bi)&=(e^b+e^{-b})\cos a,\\
\cos(a+bi)-\cos(a-bi)&=-(e^b-e^{-b})\,i\sin a.
\end{aligned}$$

Problem 7.

112. *The sine of an angle being real, but greater than the total sine, so that the angle is imaginary, to find the value of this angle.*

Solution

There are two cases here, according to whether the given sine is positive or negative.

I. So first let the positive sine be p, and $p>1$, always taking unity for the total sine, and let the imaginary angle which corresponds to this sine be

$a + bi$. In order that the sine be real, it is necessary that $a = (2\lambda - 1)\rho$, taking ρ for the mark of a right angle, and since according to §108

$$\sin((2\lambda - 1)\rho + bi) = \pm\tfrac{1}{2}(e^b + e^{-b}) = p,$$

it must be that λ is an odd number. So let $\lambda = 2\mu + 1$, and we will have

$$\sin\big((4\mu + 1)\rho + bi\big) = \tfrac{1}{2}(e^b + e^{-b}) = p.$$

Since this equation is always possible, we will derive from it

$$e^b = p \pm \sqrt{pp - 1} \quad \text{and} \quad b = \ln(p \pm \sqrt{pp - 1}).$$

Therefore the sought angle, which corresponds to the sine, p, will be

$$(4\mu + 1)\rho + i\ln(p \pm \sqrt{pp - 1}).$$

II. Let the given negative sine be $-p$ and $p > 1$, and it is necessary that λ be an even number. So let $\lambda = 2\mu$, and we will have

$$\sin\big((4\mu - 1)\rho + bi\big) = -\tfrac{1}{2}(e^b + e^{-b}) = -p.$$

From this we derive as in the preceding case

$$e^b = p \pm \sqrt{pp - 1} \quad \text{and} \quad b = \ln(p \pm \sqrt{pp - 1}).$$

Therefore the angle which corresponds to the negative sine, $-p$, will be

$$(4\mu - 1)\rho + i\ln(p \pm \sqrt{pp - 1}).$$

Q.E.D.

Problem 8.

113. *The cosine of an angle being real, but greater than the total sine which is* 1, *so that the angle is imaginary, to find the imaginary angle which corresponds to this cosine.*

Solution

Let $p > 1$ and the given cosine be equal to $+p$. Let the corresponding angle be $a + bi$, and by §109 it is clear we must have $a = 2\lambda\rho$, in order that

$$\cos(2\lambda\rho + bi) = \pm\tfrac{1}{2}(e^b + e^{-b}) = p,$$

so λ must be an even number. So let $\lambda = 2\mu$, and we will have

$$\cos(4\mu\rho + bi) = \tfrac{1}{2}(e^b + e^{-b}) = p,$$

from which we derive

$$e^b = p \pm \sqrt{pp-1} \quad \text{and} \quad b = \ln(p \pm \sqrt{pp-1}),$$

and therefore the positive cosine, p, corresponds to the angle

$$4\mu\rho + i\ln(p \pm \sqrt{pp-1}).$$

If the given cosine is negative, and equal to $-p$ when $p > 1$, it is necessary to take for λ an odd number. So let $\lambda = 2\mu + 1$, and the angle or arc which corresponds to the negative cosine, $-p$, will be

$$(4\mu + 2)\rho + i\ln(p \pm \sqrt{pp-1}).$$

Q.E.D.

Corollary

114. It is not important whether we take $p + \sqrt{pp-1}$ or $p - \sqrt{pp-1}$, since either is a positive quantity. We only have to remark that

$$\ln(p + \sqrt{pp-1}) = -\ln(p - \sqrt{pp-1})$$

so that this ambiguity reflects that which is essential to $\sqrt{-1}$.

Problem 9.

114′. *Given the imaginary sine of an angle, to find the value of the angle or imaginary arc which corresponds to it.*

Solution

Let $p + qi$ be the given imaginary sine, and the sought angle $a + bi$, so that it is necessary to have

$$\sin(a + bi) = p + qi.$$

Let us now compare this form $p + qi$ with what was found in problem 6, and we will have

$$p = \tfrac{1}{2}(e^b + e^{-b})\sin a \quad \text{and} \quad q = \tfrac{1}{2}(e^b - e^{-b})\cos a,$$

and from this we will derive

$$p\cos a + q\sin a = e^b \sin a\cos a$$

or

$$e^b = \frac{p}{\sin a} + \frac{q}{\cos a}$$

and

$$e^{-b} = \frac{p}{\sin a} - \frac{q}{\cos a}.$$

Now since $e^b e^{-b} = 1$, we will obtain

$$pp\cos^2 a - qq\sin^2 a = \sin^2 a\cos^2 a$$

or

$$pp - (pp+qq)\sin^2 a = \sin^2 a - \sin^4 a,$$

from which we derive:

$$\sin^2 a = \tfrac{1}{2}(1+pp+qq) \pm \sqrt{\tfrac{1}{4}(1+pp+qq)^2 - pp}$$

$$\sin a = \tfrac{1}{2}\sqrt{1+2p+pp+qq} \pm \tfrac{1}{2}\sqrt{1-2p+pp+qq}$$

$$\cos^2 a = \tfrac{1}{2}(1-pp-qq) \mp \sqrt{\tfrac{1}{4}(1+pp+qq)^2 - pp}\,.$$

But since $\cos 2a = 2\cos^2 a - 1$, we will have

$$\cos 2a = -pp - qq + \sqrt{1 - 2pp + 2qq + (pp+qq)^2}\,,$$

and since this expression is always real and less than unity, the angle $2a$ and therefore also a will be real, and we will find

$$\sin a = \sqrt{\frac{1-\cos 2a}{2}} \quad \text{and} \quad \cos a = \sqrt{\frac{1+\cos 2a}{2}}\,.$$

Now, since these quantities have been found, along with the angle a, we will have

$$b = \ln\left(\frac{p}{\sin a} + \frac{q}{\cos a}\right),$$

and the angle which corresponds to the sine, $p + qi$, will be

$$a + bi.$$

Q.E.D.

Corollary

115. If the given sine is simply imaginary, or equal to qi, so that $p = 0$, it is clear that we must have $\sin a = 0$, and therefore $a = 2\lambda\rho$, where ρ indicates the right angle and λ an arbitrary whole number. Then we will have

$$q = \pm\tfrac{1}{2}(e^b - e^{-b}),$$

according to whether λ is an even or odd number. So we will have

$$e^b = \pm q \pm \sqrt{qq + 1},$$

and therefore we will always be able to render this quantity positive, from which we will derive

$$b = \ln(\sqrt{qq + 1} \pm q),$$

where the sign $+$ occurs if λ is an even number, and the sign $-$ if λ is odd. Thus the arc which corresponds to the sine, qi, will be either

$$4\mu\rho + i\ln(\sqrt{qq + 1} + q)$$

or

$$(4\mu + 2)\rho + i\ln(\sqrt{qq + 1} - q).$$

Problem 10.

116. *Given the imaginary cosine of an angle, to find the imaginary value of the arc or angle which corresponds to it.*

Solution

Let $p + qi$ be the given imaginary cosine, and $a + bi$ the corresponding arc sought, so that

$$\cos(a + bi) = p + qi.$$

If we relate this equality to article 106, we will have

$$p = \tfrac{1}{2}(e^b + e^{-b})\cos a \quad \text{and} \quad q = -\tfrac{1}{2}(e^b - e^{-b})\sin a,$$

from which we will obtain

$$e^b = \frac{p}{\cos a} - \frac{q}{\sin a}$$

and

$$e^{-b} = \frac{p}{\cos a} + \frac{q}{\sin a}$$

and

$$\cos^2 a = \tfrac{1}{2}(1 + pp + qq) \pm \sqrt{\tfrac{1}{4}(1 + pp + qq)^2 - pp}\,,$$

so

$$\cos 2a = pp + qq - \sqrt{1 - 2pp + 2qq + (pp + qq)^2}\,,$$

which is also always real and less than the total sine, and from this we will have

$$\sin a = \sqrt{\frac{1 - \cos 2a}{2}} \quad \text{and} \quad \cos a = \sqrt{\frac{1 + \cos 2a}{2}},$$

and having determined the angle a itself, because of

$$b = \ln\Big(\frac{p}{\cos a} - \frac{q}{\sin a}\Big)$$

the angle or arc which corresponds to the imaginary cosine, $p + qi$, will be

$$a + bi.$$

Q.E.D.

Corollary

117. If $p = 0$, so that the given cosine is equal to qi, we will have $\cos a = 0$, and therefore $a = (2\lambda - 1)\rho$, from which we derive

$$q = -\tfrac{1}{2}(e^b - e^{-b}) \cdot \pm 1,$$

where the top sign applies if λ is an odd number, and the bottom, if even. We will then have:

$$\begin{aligned} e^{2b} &= \mp 2e^b q + 1, \\ e^b &= \mp q + \sqrt{qq + 1}, \\ b &= \ln(\sqrt{qq + 1} \mp q), \end{aligned}$$

and the arc which corresponds to the cosine, qi, will be either

$$(4\mu + 1)\rho + i\ln(\sqrt{1 + qq} - q)$$

or

$$(4\mu + 3)\rho + i\ln(\sqrt{1 + qq} + q).$$

Scholium

118. Having found the value of $\cos 2a$, if we look to it for the values

$$\sin a = \sqrt{\frac{1-\cos 2a}{2}} \quad \text{and} \quad \cos a = \sqrt{\frac{1+\cos 2a}{2}},$$

we can take them to be either positive or negative. In order to make the choice, it is necessary to look to the quantities p and q, whether they are positive or negative, and to then give to $\sin a$ and $\cos a$ the signs which render the values of e^b and e^{-b} positive, both in this problem and in the preceding. So in each case the choice is easy to make, so that we are always able to find real values for the letters a and b.

Problem 11.

119. *An imaginary tangent being given, to find the imaginary value of the angle or arc which corresponds to it.*

Solution

Let $p+qi$ be the given imaginary tangent, and $a+bi$ the arc which goes with this tangent, so that

$$\tan(a+bi) = p+qi.$$

We found above in §107 that

$$\tan(a+bi) = \frac{2e^{2b}\sin 2a + i\,(e^{4b}-1)}{e^{4b}+2e^{2b}\cos 2a + 1},$$

so it is necessary that

$$p = \frac{2e^{2b}\sin 2a}{e^{4b}+2e^{2b}\cos 2a+1} \quad \text{and} \quad q = \frac{e^{4b}-1}{e^{4b}+2e^{2b}\cos 2a+1}.$$

From this we derive these two equations:

$$\begin{aligned} e^{4b}p + 2e^{2b}(p\cos 2a - \sin 2a) + p &= 0 \\ e^{4b}(q-1) + 2e^{2b}q\cos 2a + q + 1 &= 0, \end{aligned}$$

and by elimination

$$e^{2b} = \frac{-p}{p\cos 2a + (q-1)\sin 2a} = \frac{-p\cos 2a + (q+1)\sin 2a}{p}.$$

So we will have

$$0 = pp\,(1 - \cos^2 2a) + 2p \sin 2a \cos 2a + (qq - 1) \sin^2 2a$$

or

$$0 = pp \sin 2a + 2p \cos 2a + (qq - 1) \sin 2a.$$

Consequently, we will have

$$\tan 2a = \frac{2p}{1 - pp - qq}$$

and therefore

$$\sin 2a = \frac{2p}{\sqrt{4pp + (1 - pp - qq)^2}}$$

and

$$\cos 2a = \frac{1 - pp - qq}{\sqrt{4pp + (1 - pp - qq)^2}}\,.$$

So

$$e^{2b} = \frac{pp + (1 + q)^2}{\sqrt{4pp + (1 - pp - qq)^2}}$$

and

$$b = \tfrac{1}{2} \ln(pp + (1 + q)^2) - \tfrac{1}{4} \ln(4pp + (1 - pp - qq)^2).$$

So having found by these formulas both the value for b and that of the angle $2a$ or a, the arc which corresponds to the imaginary tangent $p + qi$ will be

$$a + bi.$$

Q.E.D.

Corollary 1.

120. Since

$$4pp + (1 - pp - qq)^2 = (pp + (q + 1)^2)(pp + (q - 1)^2),$$

it will be that

$$e^{4b} = \frac{pp + (q + 1)^2}{pp + (q - 1)^2}$$

and therefore

$$b = \tfrac{1}{4} \ln \frac{pp + (q+1)^2}{pp + (q-1)^2}.$$

Now, the angle a is most conveniently determined from the formula of the tangent

$$\tan 2a = \frac{2p}{1 - pp - qq},$$

from which we see that the values of a and b will always be real.

Corollary 2.

121. If $p = 0$, or one wishes to find the angle whose tangent is qi, we will have $\tan 2a = 0$, so $2a = 2\lambda\rho$ and

$$a = \lambda\rho \quad \text{and} \quad b = \tfrac{1}{4} \ln \frac{(q+1)^2}{(q-1)^2}.$$

Consequently, to the tangent qi corresponds the arcs

$$\lambda\rho + \frac{i}{4} \ln \frac{(q+1)^2}{(q-1)^2},$$

where $\lambda\rho$ indicates an arbitrary multiple of the right angle.

Corollary 3.

122. Here the case where $q+1=0$ or $q-1=0$ requires a special reduction, which is necessary to make before setting $p = 0$. So let $qq - 1 = 0$, or the given tangent equal to $p \pm i$, and we will have

$$\tan 2a = -\frac{2}{p}$$

and

$$e^{4b} = \frac{pp + 2 \pm 2}{pp + 2 \mp 2},$$

which is to say for the top sign

$$e^{4b} = \frac{pp + 4}{pp}$$

and for the bottom

$$e^{4b} = \frac{pp}{pp + 4}.$$

Now if $p = 0$, we will have

$$2a = (2\lambda + 1)\rho$$

because of $\tan 2a = \infty$ and $b = \pm\infty$. Therefore, to the tangent $\pm i$ corresponds the angle

$$(\lambda + \tfrac{1}{2})\rho \pm \infty \cdot i.$$

Corollary 4.

123. When

$$pp = 1 - qq \quad \text{or} \quad p = \sqrt{1 - qq}$$

we will have

$$\tan 2a = \infty$$

and

$$2a = (2\lambda + 1)\rho \quad \text{or} \quad a = (\lambda + \tfrac{1}{2})\rho.$$

Then we will have

$$b = \tfrac{1}{4} \ln \frac{1 + q}{1 - q},$$

so that to the tangent $\sqrt{1 - qq} + qi$ corresponds to the arc

$$(\lambda + \tfrac{1}{2})\rho + \frac{i}{4} \ln \frac{1 + q}{1 - q}.$$

Scholium

124. So since all these imaginary quantities, which are formed by transcendental operations, are also included in the general form

$$M + Ni,$$

we may hold without hesitation that in general all imaginary quantities, however complicated they may be, are always reducible to the form $M + Ni$, or that they are always composed of two members, where one is real, and the other is a real quantity multiplied by $\sqrt{-1}$.